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INTERIOR-EXTERIOR PENALTY APPROACH FOR SOLVING ELASTO-HYDRODYNAMIC LUBRICATION PROBLEM: PART I

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Abstract. A new interior-exterior penalty method for solving quasi-variational inequality and pseudo-monotone operator arising in two-dimensional point contact problem is analyzed and developed in discontinuous Galerkin finite volume (DG-FVEM) framework. We derive a discrete DG-FVEM formulation of the problem and prove existence and uniqueness results for it. Optimal error estimates in H^1 and L^2 norm are derived under a light load parameter assumptions. In addition, the article provides a complete algorithm to tackle all numerical complexities appear in the solution procedure. Numerical outcomes are presented for light, moderate and relative high load conditions. The variations of load parameter and its effect on the evolution of deformations and pressure profile are evaluated and described. This method is well suited for solving elasto-ehydrodynamic lubrication point contact problems and can probably be treated as commercial software. Furthermore, the results give a hope for the further development of the scheme for extreme load condition, observes in a more realistic operating situation which will be discussed in part II.

Key words. Elasto-hydrodynamic lubrication, discontinuous finite volume method, interior-exterior penalty method, pseudo-monotone operators, variational inequality.

1. Introduction

In the last century, a various attempts have been devoted by a large scientific community in shaping a more solid mathematical foundation, modeling and developing a robust scientific tool in the area of lubrication theory (study of thin-film flows). In particular, elasto-hydrodynamic lubrication (EHL) has picked up a notable innovation pace since its acceptance as the essential physical phenomenon behind the flourishing operation of many important industrial devices such as journal bearings, rolling contact bearings, gears etc. An extensive list of contributions of EHL model and theoretical development can be found mainly in [6, 29, 37, 12, 11, 24, 25].

EHL is indispensable mechanism of thin fluid-film lubrication characterized by high contact pressure. As a consequence an exceptional elastic deformation and piezo-viscous increase in lubricant viscosity. The mechanical action (squeezing, shearing etc) changes the lubricants film thickness, viscosity and density which account for the variation of bearing performance characteristics. Therefore, a qualitative and precise prediction of the elasto-hydrodynamic lubrication model requires consideration of the constitutive equation for the lubricant. In literature, numerous models [21, 29] have been introduced to describe the basic aspects of the EHL theory, where the three main attributes of this kind problems are quoted; the fluid hydrodynamic displacement (govern by Reynolds equation), the solid elastic deformation and the cavitation generation. Typical lubricated devices consist of a thin flow of lubricant between two contacting geometries in relative movement. Classically, this equation (known as Reynolds equation) is described by making heuristic

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rational argument (see [35], for example) and later by deducing from Stokes equation with the help of asymptotic techniques (see Bayada and Chambat [15]).

In the current study, two important features of the lubricating fluid are density and viscosity. We consider density of lubricant as function of applied pressure (that is non-homogeneous fluid) and it is governed by the empirical relation proposed by Dowson and Higginson [14] (see eqn (6)). We examine piezoviscous property of lubricant into account, ie. the viscosity is no longer constant and it obeys the Barus law [5] (see eqn (5)). The piezoviscous regimes has led to many outcomes based on mathematical analysis that describe the existence and uniqueness of the solution (see [12, 37, 24, 25], for example), as well as to design precise numerical methods for approximating the corresponding solutions, which have no analytical illustrations. An alternate pressure-viscosity relation also has been considered by many authors in literature (see [38, 21], for example).

It is well known that cavitation is one of the crucial phenomenon in EHL problems and it is interpreted as the rupture of the continuous fluid film due to formation of air bubbles inside the region. This phenomenon has been experimentally observed in many lubricated devices such as journal-bearings, ball-bearings, etc. Different cavitation models have been suggested for physical and mathematical analysis (e.g., see [6, 29]). One common ingredient of most of these models is the decomposition of the EHL region into two parts a lubricated part where Reynolds equation is governed and a cavitation region where the lubricant pressure is constant and equal to the saturation pressure. Consequently, the boundary splitting both regions is also priori not known in the problem, therefore modeling of the cavitation was used to impose a free boundary based on the following condition,

$$u_c = \nabla u.\mathbf{n} = 0,$$

where u_c stands for the cavitation pressure and $\mathbf{n}$ stands for the unit normal vector to the free boundary. This condition leads to a formulations in terms of a complementarity problem associated with the corresponding nonlinear Reynolds equation, or equivalently to a variational inequality (see eqns (1)–(3)).

Over last few decades, renowned interest has been paid to study Elastohydrodynamic lubrication (EHL) model problems in terms of theoretical as well as practical points of view. A significant amount of numerical techniques are available in literature for the EHL model problems [30, 21, 20, 19, 23, 36, 1, 32, 42, 43, 22, 11, 40, 31, 24, 45, 41]. Recently, theoretical study of finite volume element method (FVEM) [28] and discontinuous Galerkin finite volume element method (DG-FVEM) such as optimal error analysis is gaining momentum. In particular, these methods can be derived from a firm theoretical foundation and understanding similar to finite element, see for example [25, 7, 8, 9, 27, 17, 18]. Formulation of these methods is derived by integrating the partial differential equation (PDE) model over a control volume element. Due to its natural conservation characteristic feature, adaptivity and parallelizability, DG-FVEM gained popularity in development of scientific computing software for many real life complex phenomenon such as fluid mechanics, contact problems in mechanics, mathematical finance and hyperbolic conservation laws (traffic flow etc.) where analytical solution has minimum regularity of in nature. In many cases, implementing high order methods is not straight forward and it requires huge computational storage and time. On the other hand, using low order scheme we pay the price in the form of low accuracy when discretization grid are not small. To achieve the numerical accuracy one choice we have is, refine the mesh and use the parallelization. Hence, it is quite reasonable to demand the