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THE WEAK GALERKIN FINITE ELEMENT METHOD FOR SOLVING THE TIME-DEPENDENT STOKES FLOW

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Abstract. In this paper, we solve the time-dependent Stokes problem by the weak Galerkin (WG) finite element method. Full-discrete WG finite element scheme is obtained by applying the implicit backward Euler method for time discretization. Optimal order error estimates are established for the corresponding numerical approximation in H^1 norm for the velocity, and L^2 norm for both the velocity and the pressure in semi-discrete forms and full-discrete forms, respectively. Some computational results are presented to demonstrate the accuracy, convergence and efficiency of the method.

Key words. Stokes problem, weak Galerkin finite element method, discrete weak gradient, discrete weak divergence.

1. Introduction

The Stokes problem [27] describes the dynamics of fluid flows in complex porous media. It has wide applications in industrial and scientific fields, such as, petroleum, biomedical engineering, and heat conduction model, etc. In this paper, we consider the time-dependent Stokes problem, which has been treated by various numerical methods, such as the finite element methods (FEMs) [9, 13], the finite volume methods [1, 21], the discontinuous Galerkin methods [2, 3, 10, 28], and the weak Galerkin finite element methods [4, 15]. We provide a new developed weak Galerkin finite element method in this paper. The concerning time-dependent Stokes equation seeks the velocity function $\mathbf{u}$ and pressure function p satisfying

$$\mathbf{u}_t - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \text{ in } \Omega \times (0,T], \quad (1)$$

$$\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega \times (0,T], \quad (2)$$

$$\mathbf{u} = \mathbf{g}, \text{ on } \partial\Omega \times (0,T], \quad (3)$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}^0, \text{ in } \Omega, \quad (4)$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2,3)$. $\mathbf{f}$ is a momentum source term, $\mu > 0$ is the kinematic viscosity, and $\mathbf{u}_t$ is the time partial derivation of $\mathbf{u}(x,t)$. We assume that $\mathbf{f}, \mathbf{g}$ and $\mathbf{u}^0$ are given, sufficiently smooth functions. For simplicity, we consider (1) and (3) with $\mu = 1$ and $\mathbf{g} = \mathbf{0}$.

The weak forms in the primary velocity-pressure formulations for the Stokes problems (1)-(4) find $(\mathbf{u};p) \in L^2(0,T;[H_0^1(\Omega)]^d) \times L^2(0,T;L_0^2(\Omega))$, for any $(\mathbf{v};q) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ with $t \in (0,T]$ satisfying

$$\begin{aligned}(\mathbf{u}_t,\mathbf{v}) + (\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) &= (\mathbf{f},\mathbf{v}),\\ (q,\nabla\cdot\mathbf{u}) &= 0.\end{aligned}$$

For the discretization of the Stokes equation, we use the weak Galerkin (WG) finite element method. The WG method was first introduced in 2011 [14] for the second order elliptic problem and further applied to other partial differential equations, for example, the second order elliptic equation[12, 16, 22], the Stokes equation

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[17, 18, 24], linear elasticity equations [11], the parabolic equations [22, 25, 29] the Brinkman equation [19, 23], the Biharmon problem [6, 7, 26] and the Helmholtz equation [8], etc. The WG method refers to a general finite element technique for partial differential equations in which differential operators are approximated by weak forms as distributions for generalized functions. The main idea of the WG method is the use of weak functions and their corresponding weak derivatives defined as distributions. Weak functions and weak derivatives can be approximated by polynomials with arbitrary degrees. Thus, there are three prominent features: (1) The usual derivatives are replaced by distributions or discrete approximations of distributions; (2) The approximating functions are discontinuous; (3) The WG method allows the use of finite element partitions with arbitrary shape of polygons in $2D$ or polyhedra in $3D$ with certain shape regularity. These features make the WG method have many advantages, such as high order of accuracy, high flexibility, and easy handling of complicated geometries.

In this paper, we provide an effective WG finite element method for the time-dependent Stokes equation. The weak Galerkin finite element space consists of discontinuous piecewise polynomials of degree $k \geq 1$ for the velocity $\mathbf{u}$ and polynomials of degree $k-1$ for the pressure p, respectively. The paper is organized as follows. In Section 2, we introduce some standard notations in Sobolev space and then develop the semi-discrete and full-discrete WG finite element scheme for the Stokes equation (1)-(4). For time discretization, we use the backward Euler method, which is an implicit method. In Section 3, we derive the semi-discrete and full-discrete error equations for the WG approximations. Optimal order error estimates for both the semi-discrete and full-discrete backward Euler WG finite element approximations are established in Section 4 in H^1 norm for the velocity and L^2 norm for both the velocity and the pressure functions. Finally, in Section 5, we present some numerical experiments to confirm the theoretical analysis.

2. The Weak Galerkin Finite Element Method

In this section, we introduce some preliminaries and notations for Sobolev space, the semi-discrete and full-discrete WG finite element schemes for the Stokes problem (1)-(4).

Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$ $(d = 2, 3)$. We use the standard notations for the Sobolev space $H^s(D)$, and the associated inner product $(\cdot, \cdot)_{s,D}$, norm $\|\cdot\|_{s,D}$, and semi-norm $|\cdot|_{s,D}$ for any $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in $\mathbb{R}^2$ or polyhedral in $\mathbb{R}^3$ satisfying a set of conditions [5], and T be each element with ∂T as its boundary. $\mathcal{E}_h$ is the set of all edges or flat faces in $\mathcal{T}_h$, and $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ is the set of all interior edges or flat faces. For each $T \in \mathcal{T}_h$, denote by h_T the diameter of T, and $h = \max_{T \in \mathcal{T}_h}$ is the mesh size of $\mathcal{T}_h$.

We define weak Galerkin finite element space for the velocity function $\mathbf{u}$ and the pressure function p, as follows

$$
\begin{aligned}
V_h &= \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}, \mathbf{v}_0|_T \in [P_k(T)]^d, \mathbf{v}_b|_e \in [P_{k-1}(e)]^d, \forall T \in \mathcal{T}_h, \forall e \in \partial T\}, \\
V_h^0 &= \{\mathbf{v} \in V_h, \mathbf{v}_b = 0 \text{ on } \partial\Omega\}, \\
W_h &= \{q : q \in L_0^2(\Omega), q|_T \in P_{k-1}(T), \forall T \in \mathcal{T}_h\}.
\end{aligned}
$$