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ASYMPTOTICALLY EXACT A *POSTERIORI* ERROR ESTIMATES FOR THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR KDV EQUATIONS IN ONE SPACE DIMENSION

MAHBOUB BACCOUCH

Abstract. In this paper, we develop and analyze an implicit *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method for nonlinear third-order Korteweg-de Vries (KdV) equations in one space dimension. First, we show that the LDG error on each element can be split into two parts. The first part is proportional to the $(p+1)$-degree right Radau polynomial and the second part converges with order $p + \frac{3}{2}$ in the L^2-norm, when piecewise polynomials of degree at most p are used. These results allow us to construct *a posteriori* LDG error estimates. The proposed error estimates are computationally simple and are obtained by solving a local steady problem with no boundary conditions on each element. Furthermore, we prove that, for smooth solutions, these *a posteriori* error estimates converge at a fixed time to the exact spatial errors in the L^2-norm under mesh refinement. The order of convergence is proved to be $p + \frac{3}{2}$. Finally, we prove that the global effectivity index converges to unity at $\mathcal{O}(h^{\frac{1}{2}})$ rate. Several numerical examples are provided to illustrate the global superconvergence results and the convergence of the proposed error estimator.

Key words. Local discontinuous Galerkin method, nonlinear KdV equations, superconvergence, *a posteriori* error estimation.

1. Introduction

KdV-type equations describe the propagation of waves in a variety of nonlinear, dispersive media and appear often in many physical applications; see *e.g.* [27, 30] and the references therein. In this paper, we propose and analyze a residual-based *a posteriori* error estimator for the local discontinuous Galerkin (LDG) method for one-dimensional nonlinear Korteweg-de Vries (KdV) equations of the form

$$u_t + (f(u))_x + u_{xxx} = g(x,t), \quad x \in \Omega = [a,b],\ t \in [0,T], \tag{1a}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1b}$$

and periodic boundary conditions. Here, $g(x,t)$, and $u_0(x)$ are some given smooth functions. We assume that the nonlinear flux function $f(u)$ is sufficiently smooth with respect to the variable u and the exact solution is also smooth on $[a,b] \times [0,T]$ for a fixed time T. For the sake of simplicity,

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we only consider periodic boundary conditions. This assumption is not essential and the LDG scheme can be easily designed for purely Dirichlet or Mixed Dirichlet-Neumann boundary conditions; see [4, 7, 13, 19] for some discussion.

The LDG method is a successful numerical technique for solving linear and nonlinear partial differential equations (PDEs) containing higher than first-order spatial derivatives. It was first introduced by Cockburn and Shu [29] for solving convection-diffusion problems. Since then, several LDG schemes have been developed and analyzed for various high order differential equations in one and multiple dimensions including two-point boundary-value problems [20, 21, 22, 23, 24], convection-diffusion problems [2, 4, 7, 13, 26, 29], second-order wave equations [3, 9, 10, 11, 14], the sine-Gordon equation [15, 16, 17, 18, 25], KdV-type equations [12, 19, 31, 33, 34, 35, 36, 37], and the fourth-order Euler-Bernoulli beam equation [5, 6, 8], just to mention a few. The LDG method has many advantages over the classical numerical methods available in the literature such as the finite difference and finite element methods. For instance, LDG methods are robust and high-order accurate, can achieve stability without slope limiters, and are element-wise conservative. Moreover, LDG methods are extremely flexible in the mesh-design, they can easily handle meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders. As we shall see below, they further exhibit global superconvergence properties that can be used to construct asymptotically exact *a posteriori* error estimates by solving a local residual problem on each element. More details about the LDG methods for high order time dependent PDEs can be found in the review paper [35] and the proceeding of Shu [32]. Furthermore, some LDG methods for solving high order PDEs were developed by Yan and Shu [38], which were high order accurate and stable schemes.

In [12], we presented *a posteriori* error estimates for the LDG method for the linearized KdV equation in one space dimension $u_t + \alpha u_x + \beta u_{xxx} = 0$. The proposed error estimates are computationally simple and are obtained by solving a local steady problem with no boundary condition on each element. We proved that the significant parts of the spatial discretization errors for the LDG solution and its spatial derivatives (up to second order) are proportional to $(p+1)$-degree Radau polynomials. We used these results to develop asymptotically exact *a posteriori* error estimates. We also proved that, for smooth solutions, the proposed *a posteriori* LDG error estimates for the solution and its spatial derivatives, at a fixed time t, converge to the true errors at $\mathcal{O}(h^{p+\frac{3}{2}})$ rate. The purpose of this paper is to extend these results to nonlinear KdV equations of the form (1). In [19], we presented and analyzed a superconvergent LDG scheme for solving (1). Optimal *a priori* error estimates for the LDG solution and for the two auxiliary variables that approximate the first- and second-order derivatives are derived in the L^2-norm. The order of convergence is proved to be $p+1$. We also proved that the derivative of the LDG solution is superconvergent with order $p+1$