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NUMERICAL ANALYSIS OF A HISTORY-DEPENDENT VARIATIONAL-HEMIVARIATIONAL INEQUALITY FOR A VISCOPLASTIC CONTACT PROBLEM

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Abstract. In this paper, we consider a mathematical model which describes the quasistatic frictionless contact between a viscoplastic body and a foundation. The contact is modeled with normal compliance and unilateral constraint. We present the variational-hemivariational formulation of the model and prove its unique solvability. Then we introduce a fully discrete scheme to solve the problem and derive an error estimate. Under appropriate regularity assumptions of the exact solution, we obtain the optimal order error estimate. Finally, numerical results are reported to show the performance of the numerical method.

Key words. Variational-hemivariational inequality, viscoplastic material, numerical approximation, optimal order error estimate.

1. Introduction

In this paper, we consider a frictionless contact model for rate-type viscoplastic materials. The constitutive law of such materials can be described in the form of

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{1}$$

where $\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})$ denote the displacement, the stress tensor and the linearized strain tensor, respectively. Operator $\mathcal{E}$ is linear and describes the elastic properties of the material. Operator $\mathcal{G}$ is a nonlinear constitutive function and describes the viscoplastic behavior.

Viscoplastic models are used to describe the behavior of real materials like rubbers, metals, rocks and so on. Concrete examples, experimental background and mechanical interpretation concerning viscoplastic materials can be found in [8]. Mathematical modeling, well-posedness and numerical analysis concerning (1) and its variations can be found in [24, 4, 10, 1, 25] and references therein. For comprehensive studies, we also refer to the book [13]. However, all these monographs are in the framework of variational inequalities.

The notation of hemivariational inequality was first introduced in the 1980's ([23]). It is related to the concept of the generalized gradient of a locally Lipschitz function ([7]). In contrast to variational inequalities with convex structures, hemivariational inequalities are mathematical problems involving nonconvex terms. Particularly, variational-hemivariational inequalities involve both convex and nonconvex terms. During the last three decades, hemivariational inequalities were shown to be a very useful tool, especially in contact mechanics ([21]). Various applications to viscoelastic contact models have been studied in [12, 2, 15, 14]. Moreover, if there is a history-dependent operator in the viscoelastic contact model, the problem leads to the history-dependent hemivariational inequality ([19, 26, 22, 27]).

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Compared with the well-developed studies on viscoelastic contact models, there are relatively few publications devoted to hemivariational inequalities for viscoplastic materials. The difficulty lies in the complex viscoplastic constitutive law. Taking the integral of Equation (1):

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}(0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(0)), \tag{2}$$

it naturally contains the history-dependent term. What's more, the constitutive law has an implicit expression of stress field $\boldsymbol{\sigma}$. It means that, when proving the existence result, we need to consider a coupled system which is a history-dependent hemivariational inequality combined with an integral equation, rather than only one hemivariational inequality. When deriving error estimates, since $\boldsymbol{\sigma}$ can not be described by $\boldsymbol{u}$ directly, we have to handle both $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, rather than only $\boldsymbol{u}$.

Related references are in the following. In [5], a quasistatic viscoplastic contact problem is proved to have a unique weak solution. The existence and uniqueness results are obtained for the quasistatic contact model with memory term in [16], moreover with memory and damage terms in [17]. The unique weak solvability for a dynamic contact problem is the topic of [20]. In [18], the dynamic contact problem with damage is proved to have a unique weak solution. To our knowledge, numerical analysis and numerical simulation for hemivariational inequalities for viscoplastic materials have not been investigated in the literature so far and we fill this gap in the present paper. The problem concerned here is a quasistatic contact with normal compliance, unilateral constraint and viscoplastic materials.

The paper is structured as follows. In Section 2, we present some necessary preliminaries. In Section 3, we describe the model of the contact process, derive its variational-hemivariational formulation, state the existence and uniqueness theorem and prove it. Then in Section 4, we introduce a fully discrete scheme and provide the error estimates. Finally, in Section 5, we present some numerical examples which provide numerical evidence of our theoretical results.

2. Preliminaries

In this section, we present some necessary notation and preliminary material which we will use in our paper.

Let X be a Banach space. We first recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $\varphi : X \to \mathbb{R}$ ([7]). The generalized directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x,\, t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}.$$

The generalized gradient of φ at x, denoted by $\partial_{Cl}\varphi(x)$, is a subset of a dual space X^* given by $\partial_{Cl}\varphi(x) = \{\, \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}$ for all $v \in X \}$. In particular, here we present two basic properties provided in [7]:

$$\varphi^0(x; v) = \max \{\, \langle \zeta, v \rangle \mid \zeta \in \partial_{Cl}\varphi(x) \,\}, \tag{3}$$

$$\varphi^0(x; v_1 + v_2) \leq \varphi^0(x; v_1) + \varphi^0(x; v_2). \tag{4}$$

Let d be a positive integer. The linear space of second-order symmetric tensors on $\mathbb{R}^d$ is denoted by $\mathbb{S}^d$. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\, \tau_{ij}, \qquad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \qquad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$