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A POD-BASED FAST ALGORITHM FOR THE NONLOCAL UNSTEADY PROBLEMS

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Abstract. A fast algorithm for the nonlocal unsteady problems was proposed, which can be employed in the numerical simulation of nonlocal diffusion and peridynamic. The surrogate model constructed by the proper orthogonal decomposition (POD) speeds up the process of solving equations by reducing the order of linear equations. Then, the accuracy and efficiency of the proposed algorithm was verified by numerical experiments. The results showed that this approach ensures accuracy while reduces the computational burden of the nonlocal model.

Key words. Nonlocal diffusion, peridynamic, fast algorithm, POD, model reduction.

1. Introduction

The classical theory of continuum mechanics assumes that all internal forces in the body are contact forces, which leads to mathematical models described by partial differential equations. However, the partial differential equation model cannot properly model problems evolving discontinuities, such as damage and fracture. The reason is that it assumes the displacement field is continuously differentiable. Peridynamic (PD) model [1, 2, 3, 4] is a reformulation of continuum theory, which avoids the explicit use of spatial derivatives and the internal force is considered to be a non-contact force. Consequently, PD models are particularly suitable for the representation of discontinuities in displacement fields and crack evolution in materials, its effectiveness in modeling material damages has been shown in numerical simulation of crack nucleation[5], crack propagation and branching[6, 7], phase transformations in solids [8], impact damage[9, 10], polycrystal fracture and so on. Various numerical methods for solving PD problems have been proposed and implemented, including finite difference[11], finite element [12, 13, 14, 15], quadrature, and particle-based[16] methods are successfully applied in numerical simulation of fracture and damage.

However, just like the fractional partial differential equation model[17, 18, 19], due to the nonlocal property of the nonlocal operator, the numerical methods for the PD models yield dense stiffness matrices. Consequently, two main factors restrict the efficiency of numerical simulation, one is the generation of the stiffness matrices, and the other is the computational complexity of the dense linear equations solver. Both two render the numerical simulation of nonlocal models computationally expensive.

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Seeking an efficient and accurate numerical algorithm has been one of the crucial issues in the application of nonlocal models. Extensive research has been developed on the fast algorithm for nonlocal models. A fast finite difference method for the one-dimensional space-fractional diffusion equation was developed in [20], in which the stiffness matrix is full, and it needs a Toeplitz-like matrix expansion. A fast Galerkin finite element method was developed for a one-dimensional PD model in [21], because of the stiffness matrix can be decomposed as a sum of a tridiagonal matrix and a Toeplitz matrix, then the Fast Fourier Transform has been used to accelerate the matrix-vector product, and it was extended to the discontinuous Galerkin method in [22]. The Galerkin method needs to compute more than one layer of integration in the assembly of the stiffness matrix, which constitutes a very large portion of computation time. Therefore, the fast collocation method was developed for nonlocal diffusion problems. In one dimensional case, the stiffness matrix can be decomposed as a sum of a Toeplitz matrix [23, 24] and a low-rank matrix, and the stiffness matrix was proved to be of block-Toeplitz-Toeplitz-block matrices in two-dimensional case [25]. Recently, a fast collocation method is discretized on a uniform partition for two-dimensional peridynamic problems [26], in which the stiffness matrix was also proved to be block-Toeplitz-Toeplitz-block matrices. All of these fast algorithms developed above speed up the solving by exploring the structure of the stiffness matrices, which is discretized on a uniform partition. It can be seen later, the fast algorithm proposed in this paper is independent of the mesh structure.

In this paper, a fast algorithm for the nonlocal unsteady problem was proposed by constructing the reduced-order model (ROM) of the original system. The key idea of ROM is approximating the high-order system with a lower-order system based on the proper orthogonal decomposition (POD) method [27] and the projection coefficients of ROM are obtained by implementing Galerkin projection [28]. In [29], the ROM was firstly used to solve parameterized nonlocal problems. Different from that, this method is utilized in speed up the equation solving at this time. Actually, this algorithm only focus on second factor. However, compared with the existing fast algorithm, our method is independent of the structure of stiffness matrix and it can be easily embedded into existing numerical methods. In the process of solving, it plays a significant role in reducing the computational burden of solving dense linear equations.

The rest of this paper is organized as follows. In section 2, the nonlocal parabolic equation and nonlocal wave equation are introduced. The weak form of these two equations and their finite element discretizations are given in sections 3 and 4, respectively. Section 5 contains POD-based reduced-order models of both two types of equations. Finally, numerical experiments are carried out to verify the efficiency of our fast method.