INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 1, Pages 1–18

EVALUATION OF CURVES AND SURFACES BY METHODS BASED ON DIFFERENTIAL EQUATIONS AND CAGD

JORGE DELGADO AND JUAN MANUEL PEÑA

Abstract. For the evaluation of free-form polynomial and trigonometric curves and surfaces, several Taylor methods and two new methods (DP and DT) motivated by Computer Aided Geometric Design (CAGD) have been considered. Their accuracy and computational costs are compared through numerical examples. In the polynomial case they are also compared with the most used evaluation algorithms.

Key words. Free-form curves and surfaces, evaluation, Taylor method, polynomials, trigonometric polynomials.

1. Introduction

The design of free-form curves and surfaces through a set of control points and a blending system of basis functions is a usual tool in the fields of computer aided design, computer aided manufacturing, geometric modeling or computer graphics. Among the blending bases of univariate functions, we can mention the Bernstein basis of polynomials (see Section 3), the B-splines or the rational Bernstein basis (see [9], [13]). All these bases are examples of normalized B-bases, which are the bases with optimal shape preserving properties of their corresponding spaces (see [2]). Normalized B-bases have also been found for spaces containing other types of functions, such as exponential, trigonometric or hyperbolic functions, which are also useful in many applications (cf. [10], [11], [12], [14], [16], [17], [18], [23], [26]).

The de Casteljau and the rational de Casteljau algorithms are stable algorithms for the evaluation of polynomial or rational Bézier curves and surfaces and present additional advantages over other known evaluation algorithms (cf. [9], [3]). However, for the spaces mentioned at the end of the previous paragraph, algorithms with similar advantages to those of the de Casteljau algorithm have not been found. These and other spaces useful in CAGD (computer aided geometric design) satisfy that their basis functions are the solutions of linear differential systems. In [24] it was shown that typical numerical methods for solving the differential systems (as the Taylor method or the implicit midpoint scheme) can be used to evaluate free-form curves and also to evaluate tensor-product surfaces, which can be generated by evaluating a series of isoparametric curves at a grid of points.

A first goal of our paper is the comparison of the accuracy and computational cost of several dynamic methods related with Taylor method for evaluating algebraic and trigonometric polynomials. In the polynomial case, we consider randomly generated polynomials and Wilkinson's polynomials, and we also compare these methods with the most efficient evaluation methods (see [4], [5]). A second goal of this paper is the presentation of two new evaluation methods for algebraic and trigonometric polynomials, which are called DP (direct polynomial) and DT (direct trigonometric), respectively. Both methods use the normalized B-basis of their corresponding space and perform a direct evaluation, computing the basis functions

Received by the editors March 26, 2019 and, in revised form, October 10, 2020.
2000 *Mathematics Subject Classification.* 65D17, 65D25, 65L05.

in a nested way. The nice behavior of these algorithms is illustrated by numerical examples.

The paper is organized as follows. Section 2 presents several Taylor methods for evaluation. Section 3 presents the evaluation methods for algebraic polynomials and contains the numerical examples comparing them. The corresponding evaluation methods and numerical examples for trigonometric polynomials are presented in Section 4. Finally, Section 5 summarizes the main conclusions of the paper and considers a future work.

2. Evaluations of functions by solving system of linear differential equations

Let us consider representations given by

$$f(t) = \sum_{i=0}^{n} c_{0i}\, u_i(t), \quad t \in [a,b], \tag{1}$$

where $(c_{0i})_{i=0}^{n}$ is a sequence of real numbers and $(u_0, u_1, \dots, u_n)$ is a basis of a linear space of functions Ω.

If $u_i' \in \Omega$ for all $i = 0, 1, \dots, n$, then $f(t)$ can be evaluated by solving a system of linear differential equations according to [24]. Let us recall this method. If $u_i' \in \Omega$ for all i then there exists a matrix $R = (r_{ij})_{0 \le i,j \le n}$ such that

$$\frac{d}{dt}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = R \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}. \tag{2}$$

Differentiating (1) and taking into account the previous expression we obtain

$$f'(t) = \sum_{i=0}^{n} c_{0i}\, u_i'(t) = (c_{00}, \dots, c_{0n}) \frac{d}{dt}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = (c_{00}, \dots, c_{0n}) R \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

Let us choose $(c_{ij})_{1 \le i \le n}^{0 \le j \le n}$ such that the matrix $C = (c_{ij})_{0 \le i,j \le n}$ is nonsingular. Let us define

$$X(t) := (x_0(t), x_1(t), \dots, x_n(t))^T = C\,(u_0(t), \dots, u_n(t))^T, \quad t \in [a,b]. \tag{3}$$

We can observe that $x_0(t) = f(t)$. Differentiating the previous expression and using (2) and (3) we obtain

$$X'(t) := C \frac{d}{dt}\begin{pmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} = C\,R \begin{pmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

Taking into account that C is nonsingular and denoting $A := CRC^{-1}$ we derive from the previous expression the following system of linear differential equations:

$$X'(t) = AX(t). \tag{4}$$

Let us recall that the first component of the solution of the previous system of differential equations is $x_0(t) = f(t)$. So, each method to solve numerically a system of differential equations can be used as a method to evaluate the function $f(t)$ in (1).