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A COMPACT FINITE DIFFERENCE SCHEME FOR THE FOURTH-ORDER TIME MULTI-TERM FRACTIONAL SUB-DIFFUSION EQUATIONS WITH THE FIRST DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this paper, a finite difference scheme is established for solving the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions. Using the method of order reduction, the original problem is equivalent to a lower-order system. Then the system is considered at some particular points, and the first Dirichlet boundary conditions are also specially handled, so that the global convergence of the presented difference scheme reaches $O(\tau^2 + h^4)$, with τ and h the temporal and spatial step size, respectively. The energy method is used to give the theoretical analysis on the stability and convergence of the difference scheme, where some novel techniques have been applied due to the non-local property of fractional operators and the numerical treatment of the first Dirichlet boundary conditions. Numerical experiments further validate the theoretical results.

Key words. Multi-term, fractional sub-diffusion equations, the first Dirichlet boundary conditions, stability, convergence.

1. Introduction

With the development of science and technology, fractional differential equations are widely used in scientific research and engineering applications. Many phenomena in the fields of astronomy [1], finance [2], medicine [3], physics [4], etc. can use fractional differential equations to build models. Therefore, the theoretical researches and applications of fractional differential equations have become one of the hot issues of recent concern, which has the widespread good prospects for development. Since the solutions to many fractional differential equations cannot be accurately obtained or the form of the solution is relatively complicated, the numerical results are particularly important.

When the first-order or second-order time derivatives in the classical diffusion wave equation are replaced by fractional derivatives, the fractional diffusion wave equations are obtained. In recent years, many scholars have done a lot of researches on the second-order time fractional diffusion equations. Sun and Wu [5] analyzed the truncation errors of the $L1$ numerical approximation formula by using linear interpolation for the Caputo fractional derivative and then constructed a fully discrete difference scheme for the fractional wave equations by introducing new variables to convert the original system of equations into a lower-order system. The stability and convergence of the difference scheme were proved by the energy method. Based on the previous content, the numerical results in the case of the slow diffusion system were also briefly discussed. Du, Cao and Sun [6] further proposed the high order difference method for the fractional wave equations to improve the convergence order in space to the fourth-order. Gao and Sun [7] proposed a compact difference scheme for time fractional diffusion equations, where the stability

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and unconditional convergence of the scheme were shown by defining a new inner product. By selecting $\sigma = 1 - \frac{\alpha}{2}$, Alikhanov [8] obtained the $L2 - 1_\sigma$ formula to approximate the values of Caputo derivatives at some particular points and proved that the truncation error of this formula is $O(\tau^{3-\alpha})$, with α the order of the fractional derivative. Based on this formula, the finite difference scheme for the time fractional diffusion equation was established with the convergence accuracy of order two in both time and space. Vong and Lyu [9] proposed a finite difference scheme for a time-fractional Burgers-type equation, where the highlight of the scheme was that there is no need to use iterative methods to find the approximate solutions, and the unconditional stability together with convergence were proved.

For some physical phenomena, it is often not enough to describe these phenomena by the second-order spatial derivative term, hence the fourth-order derivative term in space need be introduced. By using the finite sine transform technique, Agrawal [10] converted a fractional differential equation from a space domain to a wave number domain and obtained the solutions to fourth-order fractional diffusion-wave equations by the method of inverse Laplace and inverse finite sine transforms. Hu and Zhang [11] applied the extrapolation technique to establish a compact difference scheme for solving the fourth-order fractional diffusion wave equations, and in Ref. [12], using the method of order reduction, an implicit compact difference scheme for the fourth-order fractional diffusion-wave equations was obtained. Wei and He [13] introduced a fully discrete local discontinuous Galerkin finite element method based on a finite difference discretization in time and local discontinuous Galerkin method in space for fourth-order time fractional equations and proved its unconditional stability and convergence. Yao and Wang [14] established a finite difference scheme with global convergence order $O(\tau^2 + h^4)$ for fourth-order fractional diffusion equations with Neumann boundary conditions by the special handling of the Neumann boundary condition. Liu et al. [15] proposed a finite element algorithm for solving nonlinear time fractional diffusion equations with the fourth-order derivative term.

The fractional diffusion wave equation plays an important role in the field of anomalous diffusion, especially the case with the time multi-term fractional derivatives. It's often called the multi-term fractional diffusion-wave equation. Jiang et al. [16] used the method of separation of variables to present the analytical solutions to the multi-term time-fractional diffusion-wave equation and the multi-term time-fractional diffusion equation. Liu et al. [17] investigated two implicit numerical methods to simulate the two-term mobile/immobile time fractional diffusion equation and the two-term time fractional diffusion equation, where the predictor-corrector method to solve the multi-term time fractional diffusion equations was proposed and the strict theoretical analysis was provided. Ren and Sun [18] obtained the difference scheme for solving one-dimensional and two-dimensional multi-term time fractional diffusion-wave equations by using the $L1$ approximation for the multi-term time Caputo fractional derivatives. Gao, Alikhanov and Sun [19] considered the interpolation approximation of the multi-term fractional derivatives at some special points and established a numerical algorithm for solving time multi-term fractional diffusion equations. Wei [20] established a fully discrete scheme using local discontinuous Galerkin method in space and classical $L1$ approximation in time and proved the stability and convergence of the resultant scheme. By extending the domain of the fractional Laplacian to a Banach space and using the multivariate Mittag-Leffler function, Sin, Ri and Kim [21] obtained the analytical solutions to the multi-term fractional diffusion equation. Reutskiy [22] introduced