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A P_2-P_1 PARTIALLY PENALIZED IMMERSED FINITE ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS

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Abstract. In this article, we develop a Taylor-Hood immersed finite element (IFE) method to solve two-dimensional Stokes interface problems. The $\mathcal{P}_2$-$\mathcal{P}_1$ local IFE spaces are constructed using the least-squares approximation on an enlarged fictitious element. The partially penalized IFE method with ghost penalty is employed for solving Stoke interface problems. Penalty terms are imposed on both interface edges and the actual interface curves. Ghost penalty terms are enforced to enhance the stability of the numerical scheme, especially for the pressure approximation. Optimal convergences are observed in various numerical experiments with different interface shapes and coefficient configurations. The effects of the ghost penalty and the fictitious element are also examined through numerical experiments.

Key words. Stokes interface problem, immersed finite element method, fictitious element, least-squares.

1. Introduction

In this paper, we consider the steady-state Stokes interface problem in the two-dimensional case. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain that is separated into Ω^+ and Ω^- by a smooth interface curve Γ. Consider the following Stokes equation in the velocity-pressure-stress form

$$-\nabla \cdot \sigma(\mathbf{u}, p) \; = \; \mathbf{f}, \text{ on } \; \Omega^+ \cup \Omega^-, \tag{1a}$$

$$\nabla \cdot \; \mathbf{u} \; = \; 0, \text{ on } \; \Omega^+ \cup \Omega^-, \tag{1b}$$

$$\mathbf{u} \; = \; \mathbf{0}, \text{ on } \; \partial\Omega. \tag{1c}$$

Here, $\mathbf{f}$ is given body force. $\mathbf{u}$ represents flow velocity field of an incompressible fluid motion, and p denotes the pressure. $\sigma(\mathbf{u}, p)$ is the stress tensor defined by

$$\sigma(\mathbf{u}, p) = 2\nu\boldsymbol{\epsilon}(\mathbf{u}) - p\mathbf{I} \tag{1d}$$

where the strain tensor $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t)$. Across the interface Γ, the viscosity coefficient $\nu(\mathbf{x})$ is discontinuous. Without loss of generality, we assume that ν is a piecewise constant function as follows:

$$\nu = \begin{cases} \nu^+, & \text{in } \; \Omega^+, \\ \nu^-, & \text{in } \; \Omega^-. \end{cases} \tag{1e}$$

Across the interface Γ, the following jump conditions are enforced:

$$[\![\mathbf{u}]\!]_\Gamma \; = \; \mathbf{0}, \text{ on } \Gamma, \tag{1f}$$

$$[\![\sigma(\mathbf{u}, p)\mathbf{n}]\!]_\Gamma \; = \; \mathbf{0}, \text{ on } \Gamma, \tag{1g}$$

where the jump $[\![\mathbf{w}(\mathbf{x})]\!]_\Gamma = \mathbf{w}^+(\mathbf{x})|_\Gamma - \mathbf{w}^-(\mathbf{x})|_\Gamma$, and $\mathbf{n}$ is the unit normal vector on Γ pointing from Ω^- to Ω^+.

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The Stokes equation is a linearization of the well-known Navier-Stokes equation. Stokes interface problems often describe multiphase flow with jumps in velocity, pressure, and physical parameters. Simulations of multiphase flow are widely applied in fields of fluid dynamics and biology. Examples of these applications include water-oil flow, bubble column reactors, drug delivery, treatment of lung diseases, and polymer blending and polymer electrolyte membrane fuel cell [28], etc.

PDE Interface problems, including aforementioned Stokes interface problem, have attracted great attention among mathematicians, computational scientists and engineers in the past decades. A wide variety of numerical methods, particularly finite element method (FEM), have been developed and matured for solving interface problems. There are roughly two classes of FEM when it comes to interface problems, namely the fitted-mesh FEM and the unfitted-mesh FEM. The fitted-mesh method, such as the conventional FEM, requires the solution mesh to be aligned with the interface; otherwise, the convergence of the numerical method could be compromised. However, this body-fitting restriction limits its applicability from problems involving a moving interface, as the solution mesh needs to be regenerated at each time level. On the contrast, unfitted-mesh methods usually alleviate or even eliminate the restriction on mesh. Structured meshes, such as Cartesian meshes, are usually adopted to solve interface problems with nontrivial interface shape. See Figure 1 for an illustration of a comparison of an unfitted Cartesian mesh and a fitted-mesh with a circular interface. This property is particularly advantageous for moving interface problems [5, 27, 30]. Numerical methods falling to this class include generalized FEM [7], extended FEM (XFEM) [15], CutFEM [23] and immersed FEM (IFEM) [34], to name only a few.

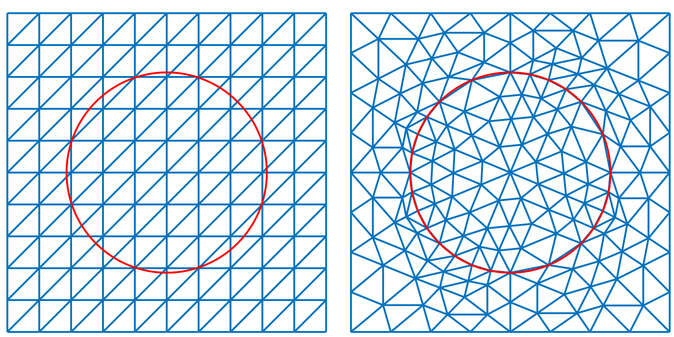

FIGURE 1. Non-body fitting (left) and body-fitting (right) meshes.

The idea of immersed finite element method [34] is to locally modify the standard FEM basis functions around the interface to satisfiy specific interface jump conditions from the physical laws. Piecewise polynomials are developed as new basis functions on all elements intersected by interfaces. Several literatures [11, 12, 26, 27, 31, 38, 44] expand this idea to multi-dimensional elliptic interface problems and higher-order approximation. Due to the discontinuity of IFE functions across the element boundaries, a partially penalized immersed finite element method (PPIFEM) was proposed in [36] as an improvement of classical IFEM. The authors added penalty terms on interface-intersected edges to the IFE scheme to enhance its stability. Many research papers on IFEM follow this idea in the recent years. For