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LOCALLY CONSERVATIVE SERENDIPITY FINITE ELEMENT SOLUTIONS FOR ELLIPTIC EQUATIONS

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Abstract. In this paper, we post-process an eight-nodes-serendipity finite element solution for elliptic equations. In the post-processing procedure, we first construct a *control volume* for each node in the serendipity finite element mesh, then we enlarge the serendipity finite element space by adding some appropriate element-wise bubbles and require the novel solution to satisfy the local conservation law on each control volume. Our post-processing procedure can be implemented in a parallel computing environment and its computational cost is proportional to the cardinality of the serendipity elements. Moreover, both our theoretical analysis and numerical examples show that the postprocessed solution converges to the exact solution with optimal convergence rates both under H^1 and L^2 norms. A numerical experiment for a single-phase porous media problem validates the necessity of the post-processing procedure.

Key words. Postprocessing, serendipity finite elements, local conservation laws, error estimates.

1. Introduction

The serendipity family of finite elements is one of the most popular finite element methods (FEMs) applying parallelepiped meshes. Over each such a mesh, the serendipity finite element space with C^0 continuity has significantly smaller dimension than the alternative tensor product Lagrange element space, yet they have the same convergence orders. With this advantage and others, the serendipity finite elements have been studied by many researchers such as Ahmad [1] in 1969, Zienkiewicz [41] in 1977, Macneal and Harder [26] in 1992, Lee and Bathe [17] in 1993, Kikuchi [15] in 1994, Kikuchi, Okabe and Fujio [16] in 1999, Zhang and Kikuchi [35] in 2000, Arnold, Boffi and Falk [3] in 2002, Rajendran and Liew [28] in 2003, Li et al. [19] in 2004, Cen and coauthors [12] in 2010, Arnold and Awanou [2] in 2011, and Rand, Gillette and Bajaj [29] in 2014, and so on.

Compared with the FEM, the finite volume method (FVM) ensures the local conservation law, which makes it widely used in scientific and engineering computations, see [4, 18, 27, 31] for an incomplete references. The finite volume element method (FVEM) is a member of FVM. The mathematical development of linear FVEM on triangular mesh is almost as satisfactory as linear FEM, see [20, 22, 36] and the references cited therein. For the bilinear FVE scheme on quadrilateral mesh, most existing works need the quasi-parallel quadrilateral assumption (e.g., [21, 23, 24, 37]), and recently [14] present a sufficient condition which covers the traditional $h^{1+\gamma}$-parallelogram mesh condition to guarantee the coercivity result.

However, it is still a challenging task to construct and analyze high order FVE schemes with optimal convergence orders, e.g., [8, 9, 20, 23, 25, 32–34, 37]. In particular, for the second order scheme, under the $h^{1+\gamma}$-parallelogram mesh assumption, [23, 37] bring the uniform stability and optimal convergence rates under both H^1 and L^2 norms. On the triangular meshes, the unconditionally stable quadratic scheme in [42] does not guarantee the optimal convergence rate in L^2 norm. Under the minimum angle condition 1.42° [39, 40], the second order scheme in [32]

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owns the optimal L^2 norm error estimate. Recently, [38] presents a class of bubble enriched quadratic schemes such that the convergence order of L^2 error is 3, regrettably the unconditionally stable of these schemes are not proved. In summary, the construction and analysis of second order FVE schemes are not easy, and the research of serendipity FVE scheme is little.

In this paper, we study the serendipity finite element method in a way different from the aforementioned works. Precisely, we post-process the eight-nodes-serendipity finite element solution so that it satisfies a desired property–the conservation law in element level. For this purpose, we first enlarge the classic serendipity finite element space by adding eight bubbles on each element. Then we construct the associated *control volumes*, each of which is a polygon surrounding a *node* of the mesh constructed by dividing each parallelepiped element with two special points on each edge and five special points (whose locations depend on the same single parameter) in the interior of the element. At the end, by solving an 8-by-8 linear system on each element, we devise our post-processed solution such that the conservation law holds on each *control volume*. Since the bubbles are the polynomials in the interior of each element and vanish at the edge of the element, the post-processed solution is globally continuous in the whole domain. More importantly, not only the post-processed solution guarantees the element-level conservation law, but also makes sure of the optimal convergence rates under both H^1 and L^2 norms.

In comparison with the FVEM, we demonstrate the importance of our results in what follows. Through comparing with the second order scheme in [37], we can see clearly that: 1) the linear system derived from the eight nodes serendipity element is symmetric, while that derived from the second order scheme in [37] is unsymmetric; 2) under the same mesh, the degree of the freedoms of the eight nodes serendipity element is much less than that of the scheme in [37]; 3) our post-processing procedure is performed on each element independently, and thus can be implemented in a parallel computing environment. In summary, directing at the high computing cost and the difficulties resided in devising and analyzing the high order FVEMs, the techniques presented here supply a better option for producing the local conservative solution and owning the optimal convergence rates. Note that the relevant works on the postprocessing technique for Lagrange finite element solution can be found in [7, 13, 43] and the references cited therein.

The rest of this paper is organized as follows. In the next section we introduce the serendipity finite elements and associated control volumes. In Section 3 we postprocess the serendipity finite element solution such that it satisfies the local conservation law on each control volume. The property that the post-processed solution owns the optimal convergence rate will be rigourously proved in this section. Numerical examples are presented in Section 4 to validate our theoretical findings. Some concluding remarks are given in Section 5.

To avoid repetition, we write "$A \lesssim B$" meaning that A can be bounded by B with a constant multiple irrelative to the parameters which A and B may depend on, while "$A \gtrsim B$" meaning that B can be bounded by A. "$A \sim B$" indicates that "$A \lesssim B$" as well as "$B \lesssim A$".

2. The serendipity finite elements and associated control volumes

We consider the finite element method for the elliptic model problem:

$$-\nabla \cdot (\beta \nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$