INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 1, Pages 38–61

MULTI-SCALE NON-STANDARD FOURTH-ORDER PDE IN IMAGE DENOISING AND ITS FIXED POINT ALGORITHM

ANIS THELJANI

Abstract. We consider a class of nonstandard high-order PDEs models, based on the $(p(\cdot), q(\cdot))$-Kirchhoff operator with variable exponents for the image denoising problem. We theoretically analyse the proposed non-linear model. Then, we use linearization method based on a fixed-point iterative technique and we also prove the convergence of the iterative process. The model has a multiscale character which follows from an adaptive selection of the exponents $p(\cdot)$ and $q(\cdot)$. The latter task helps to capture, highlight and correlate major features in the images and optimize the smoothing effect. We use Morley finite-elements for the numerical resolution of the proposed model and we give several numerical examples and comparisons with different methods.

Key words. High-order PDEs, fixed point method, anisotropic diffusion, finite elements, image restoration, inverse problems

1. Introduction

Image restoration is a fundamental task in image processing and it arises in diverse fields such as geophysics, optics, medical imaging[33, 35, 37]. It is a classical inverse problem which aims at reconstructing an image $u : \Omega \to \mathbb{R}$ from an observed one $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ that is degraded and contaminated by noise. The degradation model that we consider is the following:

$$f = u + \eta, \tag{1}$$

where η is Gaussian noise. Estimating u from the model (1) is an ill posed inverse problem where a prior image model $\mathcal{R}(u)$ is required in order to successfully estimate u from the observations f. To incorporate a prior image model $\mathcal{R}(u)$ into (1), variational approach is usually used and it consists in solving a minimization problem that have the following form:

$$\min_u \left\{ \mathcal{J}(u) := \mathcal{R}(u) + \tfrac{\lambda_0}{2} \|u - f\|^2_{L^2(\Omega)} \right\}. \tag{2}$$

The prior $\mathcal{R}(\cdot)$ in the energy $\mathcal{J}(\cdot)$ have a regularization effect and usually contains information about the image derivatives to reduce the noise that is considered as high oscillations. The second part of the energy $\mathcal{J}(\cdot)$ is the fitting term, λ_0 is a positive regularization parameter which controls the trade-off between the two terms.

A main issue in image denoising is how to choose the "best" regularization term $\mathcal{R}(\cdot)$ that can selectively smooth a noisy image without losing significant features such as edges and thin structures. Various regularizers based on first- or/and second-order derivatives have been used [14, 7, 10, 40, 25]. In [33], the authors proposed to use the well-known total variation (TV) regularizer $\mathcal{R}(u) = TV(u)$ where

$$TV(u) := \int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} \varphi dx | \varphi \in C^2_c(\Omega, \mathbb{R}^2),\ |\varphi| \le 1 \right\},$$

Received by the editors October 10, 2020.
2000 *Mathematics Subject Classification.* 65M32, 47J25, 65M50, 65M22, 94A08, 35G30, 35Q68, 65J15.

which produces a piecewise constant restored images. However, TV also produces staircase effects which is undesirable. This shortcoming gave rise to a class of a combined first- and second-order derivatives as regularizer that in general damp the noise faster and diminish the staircase effect. There have been many efforts to improve the robustness and to reduce the staircasing effects of TV using the high-order TV and total generalized variation (TGV) regularizer [11, 14]. Most of the high-order models aim to extend the works in [12] (see also, e.g., [31, 38, 39, 43]) which uses straightforward convex combinations of first- and second- derivatives. They are generally written in following form:

$$\int_\Omega G_1(\nabla u)\,dx + \int_\Omega G_2(\nabla^2 u)\,dx + \tfrac{\lambda_0}{2}\|u-f\|^2_{L^2(\Omega)}, \tag{3}$$

where $G_1(\cdot)$ and $G_2(\cdot)$ are given functions. In [41], a high-order total variation model, called $TV - TV^2$, was proposed and it consists in minimizing the following energy:

$$\alpha TV(u) + \beta TV^2(u) + \tfrac{\lambda_0}{2}||u-f||^2_{L^2(\Omega)}, \tag{4}$$

where α and β are non-negative regularization parameters chosen empirically, $TV(u)$ and $TV^2(u)$ are the total variations of u and ∇u, respectively.

Various variations of high-order models that are based on the above two energies forms were proposed [44, 48, 26, 28]. Most of these models gave rise to a second- or high-order non-linear PDEs that only consider nonlinear diffusion to denoise the image. However, nonlinear diffusion is not always the best choice for homogeneous regions, i.e. no edges but only some noise. In these regions, using linear diffusion is more appropriate as it damps noise better than nonlinear diffusion. Ideally, there should be a compromise between linear diffusion PDEs which are more interesting and effective in homogeneous regions, and nonlinear diffusion PDEs that are more powerful in regions containing edges and details.

Another class of approaches, known as nonstandard PDEs with $p(\cdot)$-growth conditions were also considered in several works (see e.g., [5, 46, 24, 36, 32]). In these approaches, the regularizer takes the form of

$$\mathcal{R}(u) = \int_\Omega |\nabla u|^{p(x)} dx,$$

where $1 \le p \le 2$. The two extreme values of the exponent $p = 1, 2$ in the regularization term lead to nonlinear (selective) diffusion and linear (isotropic) diffusion equations. In fact, the total variation model is obtained for $p = 1$ and which leads to a nonlinear diffusion PDEs where the diffusion is guided by the term $\frac{1}{|\nabla u|}$. Thus the diffusion will be selective and inverse proportional to $|\nabla u|$, i.e. for edges where $|\nabla u|$ is high, the diffusion will be enabled in order to keep edges, whereas for the homogeneous regions where $|\nabla u|$ is small, the diffusion will be strong and the model behaves similarly to a Laplace smoothness operator. For $p = 2$, the model leads to a PDE that uses the Laplace $\Delta\cdot$ as diffusion operator. The latter has an isotropic and linear diffusion property that can't distinguish between edges and homogeneous regions.

In these nonstandard regularizations, a compromise between fast/slow diffusion is made by varying $p(\cdot)$ according to the local scales. The linear diffusion is encouraged away from the edges of the image and a nonlinear correction is enforced near these singularities (see [27, 28, 26, 1, 8, 13]). To incorporate the singularity information into the $p(\cdot)$, the authors in [8] used a variable exponent $p(\cdot)$ ranging from 1 to 2 by taking $p(|\nabla u|)$ where $p(\cdot)$ is a monotone decreasing function such