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WEAKLY REGULAR STURM-LIOUVILLE PROBLEMS: A CORRECTED SPECTRAL MATRIX METHOD

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Abstract. In this paper, we consider weakly regular Sturm-Liouville eigenproblems with unbounded potential at both endpoints of the domain. We propose a Galerkin spectral matrix method for its solution and we study the error in the eigenvalue approximations it provides. The result of the convergence analysis is then used to derive a low-cost and very effective formula for the computation of corrected numerical eigenvalues. Finally, we present and discuss the results of several numerical experiments which confirm the validity of the approach.

Key words. Sturm-Liouville eigenproblems, spectral matrix methods, Legendre polynomials, acceleration of convergence.

1. Introduction

Recently, the author studied a corrected spectral matrix method for solving weakly regular and singular Sturm-Liouville problems defined over the bounded domain $(-1,1)$ with an unbounded potential at the left endpoint, [13]. The numerical results provided by such technique are definitely satisfactory for weakly regular problems. This suggested to study a generalization of the method for the approximation of the eigenvalues and of the eigenfunctions of problems of the following type

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad x \in (-1,1)\,, \tag{1}$$

$$\alpha_L y(-1) + \beta_L y'(-1) = 0\,, \qquad \alpha_L^2 + \beta_L^2 \neq 0, \tag{2}$$

$$\alpha_R y(1) + \beta_R y'(1) = 0\,, \qquad \alpha_R^2 + \beta_R^2 \neq 0, \tag{3}$$

where the potential q is given by

$$q(x) = \sum_{i=1}^{S} \frac{g_i(x)}{(1-x)^{\beta_i}(1+x)^{\gamma_i}}, \qquad \beta_i, \gamma_i < 1, \quad i = 1,\ldots,S, \tag{4}$$

with functions g_i at the numerators that are analytical inside and on a Bernstein ellipse containing $[-1,1]$. In the literature, problems of this type with q unbounded at least at one endpoint are sometimes called weakly regular and it is well known that their spectrum is composed by real and simple eigenvalues which can be ordered as an increasing sequence tending to infinity. We will number them starting from index $k = 1$, i.e. we will call

$$\{\lambda_1 < \lambda_2 < \lambda_3 < \ldots\}$$

the exact spectrum of (1)–(4).

Before proceeding, it is to be said that Sturm-Liouville eigenproblems have many applications in physics, chemistry, biology, mechanics, and so on as described, for example, in [14, 20]. Their numerical solution has been studied extensively and many schemes/codes are available nowadays (see [1, 2, 3, 4, 5, 6, 8, 11, 12, 15, 16, 17, 18, 22, 23, 24] and references therein, to mention just a few).

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Now, regarding problems with a potential function of the form specified in (4), in [13] we considered only the case $S = 2$ with $\beta_1 = \gamma_1 = \beta_2 = 0$, namely problems with a potential of the form $q(x) = g_1(x) + g_2(x)/(1+x)^{\gamma_2}$, and a special algorithm for $\gamma_2 \in (0, 1)$ and $y(-1) \neq 0$ was derived. As remarked in the same paper, the results obtained appear to be competitive with those given by other well-known schemes based on shooting techniques, [4, 11, 12, 15]. A possible explanation may be that we did not need to use a layer for handling the unbounded (but integrable) potential at the left endpoint. Concerning alternative matrix methods based on a spectral collocation approach, it must be said that a number of them, like the ones studied in [5, 8, 18, 24], refer to problems subject to the Dirichlet condition at both the endpoints.

These considerations justify the interest in generalizing the method proposed in [13] and the outline of this paper is the following. In Section 2, we recall the basic facts concerning the spectral Legendre-Galerkin matrix method introduced in [13] and we discuss the computation of the coefficient matrix that corresponds to a potential q of the form in (4). An analysis of the error in the numerical eigenvalues with respect to the generalized eigenvalue problem size is carried out in Section 3. In addition, in the same section, we derive a low cost and effective procedure for an a posteriori correction of the numerical eigenvalues. Finally, in Section 4 we report and discuss the results of some numerical experiments.

2. Spectral Legendre-Galerkin method

Let Π_{N+1} be the space of polynomials of maximum degree $N + 1$, for a fixed $N \in \mathbb{N}$, and let

$$(5)\quad \mathcal{S}_N \equiv \{r \in \Pi_{N+1} : \quad \alpha_L\, r(-1) + \beta_L\, r'(-1) = \alpha_R\, r(1) + \beta_R\, r'(1) = 0\}$$

$$(6)\quad \equiv \operatorname{span}(\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{N-1}).$$

We look for an approximation of an eigenfunction y of the following type

$$(7)\qquad z_N(x) = \sum_{n=0}^{N-1} \zeta_{n,N} \mathcal{R}_n(x) \approx y(x)$$

where the coefficients $\zeta_{n,N}$ and the numerical eigenvalue $\lambda^{(N)}$ are determined by imposing, see (1),

$$(8)\qquad \sum_{n=0}^{N-1} \left\langle \mathcal{R}_m, -\mathcal{R}_n'' + (q - \lambda^{(N)})\mathcal{R}_n \right\rangle \zeta_{n,N} = 0, \quad \text{for each } m = 0, \ldots, N-1.$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product in $L_2([-1, 1])$, i.e.

$$\langle u, v \rangle = \int_{-1}^{1} u(x)v(x)dx, \qquad u, v \in L_2([-1, 1]),$$

which is naturally suggested by the Liouville normal form of the SLP we are studying. We can write (8) as the following generalized eigenvalue problem

$$(9)\qquad (A_N + Q_N)\,\boldsymbol{\zeta}_N = \lambda^{(N)} B_N \boldsymbol{\zeta}_N$$

where $\boldsymbol{\zeta}_N = (\zeta_{0N}, \ldots, \zeta_{N-1,N})^T$,

$$(10)\qquad A_N = (a_{mn}), \quad B_N = (b_{mn}), \quad Q_N = (q_{mn}), \quad m, n = 0, \ldots, N-1,$$

with

$$(11)\qquad a_{mn} = -\langle \mathcal{R}_m, \mathcal{R}_n'' \rangle, \quad b_{mn} = \langle \mathcal{R}_m, \mathcal{R}_n \rangle, \quad q_{mn} = \langle \mathcal{R}_m, q\, \mathcal{R}_n \rangle.$$