INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 2, Pages 203–234

APPROXIMATIONS BY MINI MIXED FINITE ELEMENT FOR THE STOKES-DARCY COUPLED PROBLEM ON CURVED DOMAINS

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Abstract. In this work we solve a Stokes-Darcy coupled problem in a plane curved domain using curved elements. We approximate the velocity-pressure pair by applying the MINI-element method, for the whole coupled problem. We show that, under appropriate assumptions about the curved domain, the proposed method has optimal accuracy, with respect to solution regularity, and has a simple implementation. We also present numerical tests which show the good performance of the proposed method.

Key words. Stokes-Darcy problem, mixed finite elements, curved elements, stability analysis.

1. Introduction

The aim of this paper is to introduce and analyze a finite element scheme for solving the Stokes-Darcy coupled problem on curved domains by using curved elements. These elements are suitable for use along the curved part of the boundary and the curved part of the interface Γ. There are a wide number of papers devoted to the numerical resolution of the Stokes-Darcy coupled problem (see, for example, [4, 24, 25, 29, 30, 32] and the references therein). However, to the authors' knowledge, all analysis for the approach are restricted, in general, to the case of polygonal domains and polygonal interface or by replacing the curved domain Ω with a polygonal domain Ω_h. In [34] the authors consider curved interface and work on the interface with a "coarse scale" allowing for the grids, of the Stokes and Darcy regions, to be non-matching across interfaces. We note that, ignoring the difference between the domain Ω and its polygonal approximation Ω_h or ignoring the difference between the interface Γ_I and its polygonal approximation $\Gamma_{I,h}$, we could introduce an error which can not be compensated by an accurate approximation in the polygonal domain. Taking this observation in mind, different finite element techniques have been developed to deal with curved domains by considering curved finite elements that fit exactly the boundary (see, for example, Bernardi [9], Ciarlet and Raviart [22], Scott [33] and Zlâmal [37]). In this work, we consider curved element like those introduced in [37], which are suitable for the curved part of the interface or the boundary of the whole domain under consideration.

In the recent paper [4] the authors propose an alternative formulation of the coupled problem which allows them, in particular, to use the classical MINI elements obtaining optimal order of approximation. In this paper we generalize the ideas introduced in [4] to solve the Stokes-Darcy coupled problem in curved domains by using curved triangles which fit the curved part of the domain. We prove that, under appropriate assumptions of the curved domain, our finite element formulation satisfies the discrete inf-sup conditions, obtaining as a result optimal accuracy with respect to solution regularity. It is important to point out that our ideas could be extended to other families of elements. We focused on the MINI elements since it

Received by the editors December 28, 2019.

2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30, 74F10, 74S05.

is one of the simplest, lowest order and straightforward implementations that we could consider by using the same continuous finite element for the Stokes and Darcy equations. Numerical experiments are also presented, which confirm the excellent stability and optimal performance of our method.

The rest of the paper is organized as follows. In Section 2 we state the modified coupled Stokes-Darcy problem in a curved domain. Section 3 is devoted to describe the curved elements under consideration. In Section 4 we present the finite element approximation of the modified Stokes-Darcy problem. Finally, in section 5, we present numerical examples.

2. Problem statement

We consider a bounded open domain $\Omega \subset \mathbb{R}^2$ divided into two open subdomains Ω_S and Ω_D, where the indices S and D stand for fluid and porous regions, respectively. We assume that $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_D$, $\Omega_S \cap \Omega_D = \emptyset$ and $\overline{\Omega}_S \cap \overline{\Omega}_D = \Gamma_I$ so, Γ_I represents the interface between the fluid and the porous medium. The remaining parts of the boundaries are denoted by $\Gamma_S = \partial\Omega_S \setminus \Gamma_I$ and $\Gamma_D = \partial\Omega_D \setminus \Gamma_I$, as illustrated in Figure 1. We suppose that Γ_I, Γ_S and Γ_D are piecewise smooth Lipschitz boundaries, more precisely, that Γ_I, Γ_S and Γ_D belongs to piecewise C^{k+1} with $k \geq 1$ sufficiently large to fulfill our requirements.

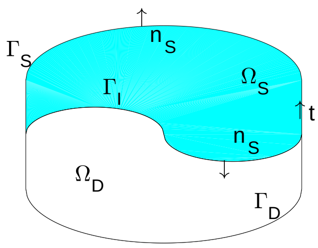

FIGURE 1. Example of two-dimensional curved domain Ω.

We denote by $\boldsymbol{n}_S$ the unit outward normal direction on $\partial\Omega_S$ and by $\boldsymbol{n}_D$ the normal direction on $\partial\Omega_D$, oriented outward. On the interface Γ_I, we have $\boldsymbol{n}_S = -\boldsymbol{n}_D$.

The Stokes-Darcy coupled problem describes the motion of an incompressible viscous fluid occupying a region Ω_S which flows across the common interface through a porous medium living in another region Ω_D saturated with the same fluid. The mathematical model of this problem can be defined by two separate set of equations and a set of coupling terms.

For any function $\mathbf{v}$ defined in Ω, taking into account that its restriction to Ω_S or to Ω_D could play different mathematical roles (especially their traces on Γ_I), we define $\mathbf{v}_S = \mathbf{v}|_{\Omega_S}$ and $\mathbf{v}_D = \mathbf{v}|_{\Omega_D}$.

In Ω_S, the fluid motion is governed by the Stokes equations for the velocity $\mathbf{u}_S$ and the pressure p_S:

$$(1) \qquad \begin{cases} -\mu\Delta\mathbf{u}_S + \nabla p_S = \mathbf{f}_S, \text{ in } \Omega_S, \\ \text{div } \mathbf{u}_S = 0, \text{ in } \Omega_S, \\ \mathbf{u}_S = 0, \text{ in } \Gamma_S, \end{cases}$$

where $\mathbf{f}_S \in (L^2(\Omega_S))^2$ represents the force per unit mass and $\mu > 0$ the viscosity.