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WELL-POSEDNESS AND THE MULTISCALE ALGORITHM FOR HETEROGENEOUS SCATTERING OF MAXWELL'S EQUATIONS IN DISPERSIVE MEDIA

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Abstract. This paper discusses the well-posedness and the multiscale algorithm for the heterogeneous scattering of Maxwell's equations in dispersive media with a periodic microstructure or with many subdivided periodic microstructures. An exact transparent boundary condition is developed to reduce the scattering problem into an initial-boundary value problem in heterogeneous materials. The well-posedness and the stability analysis for the reduced problem are derived. The multiscale asymptotic expansions of the solution for the reduced problem are presented. The convergence results of the multiscale asymptotic method are proved for the dispersive media with a periodic microstructure. A multiscale Crank-Nicolson mixed finite element method (FEM) is proposed where the perfectly matched layer (PML) is utilized to truncate infinite domain problems. Numerical test studies are then carried out to validate the theoretical results.

Key words. Maxwell's equations, dispersive medium, well-posedness, the multiscale asymptotic expansion, finite element method.

1. Introduction

Consider the transient electromagnetic wave incident on a three dimensional dispersive media with a periodic microstructure or many subdivided periodic microstructures, which is called the scatter and is supposed to occupy the bounded domain $\Omega \subset \mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz polyhedral convex domain or a bounded smooth domain with a microstructure as shown in Fig. 1(a). The exterior of the volume Ω is denoted by $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$.

Suppose that $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$ is a plane wave incident on the scatter to generate scattered field $(\mathbf{E}^{sc}, \mathbf{H}^{sc})$, which satisfies the following time-domain Maxwell's equations in Ω_e for $t > 0$:

$$(1) \qquad \begin{cases} \eta_0 \partial_t \mathbf{E}^{sc}(\mathbf{x}, t) - \mathbf{curl}\, \mathbf{H}^{sc}(\mathbf{x}, t) = 0, \\ \mu_0 \partial_t \mathbf{H}^{sc}(\mathbf{x}, t) + \mathbf{curl}\, \mathbf{E}^{sc}(\mathbf{x}, t) = 0, \end{cases}$$

where η_0 and μ_0 are the constant permittivity and the constant permeability in the "air region" Ω_e, respectively. It is clear to note that $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$ also satisfies the equation (1). In addition, the scattered field is required to satisfy the Silver-Müller radiation conditions:

$$(2) \qquad \hat{\mathbf{x}} \times (\partial_t \mathbf{E}^{sc} \times \hat{\mathbf{x}}) + \hat{\mathbf{x}} \times \partial_t \mathbf{H}^{sc} = o(|\mathbf{x}|^{-1}), \text{ as } |\mathbf{x}| \to \infty,\ t > 0,$$

where $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$. The total field $(\mathbf{E}^{tot}, \mathbf{H}^{tot})$ in Ω_e consists of the incident field and the scattered field:

$$(3) \quad \mathbf{E}^{tot}(\mathbf{x}, t) = \mathbf{E}^{inc}(\mathbf{x}, t) + \mathbf{E}^{sc}(\mathbf{x}, t),\ \mathbf{H}^{tot}(\mathbf{x}, t) = \mathbf{H}^{inc}(\mathbf{x}, t) + \mathbf{H}^{sc}(\mathbf{x}, t),\ t > 0.$$

In this paper, we investigate the well-posedness and the multiscale algorithm for the heterogeneous scattering of Maxwell's equations in dispersive media with a periodic

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microstructure or with many subdivided periodic microstructures. For the sake of simplicity, we only discuss the corresponding problems in dispersive media with a periodic microstructure in the sequel. Let $\mathbf{E}_\varepsilon(\mathbf{x},t)$ and $\mathbf{H}_\varepsilon(\mathbf{x},t))$ be respectively the electric field and the magnetic field in the scatter, which satisfy Maxwell's equations for the time-domain Lorentz model in Ω, for $t>0$:

$$
\begin{cases}
\boldsymbol{\eta}_\varepsilon(\mathbf{x})\partial_t\mathbf{E}_\varepsilon(\mathbf{x},t) - \mathbf{curl}\ \mathbf{H}_\varepsilon(\mathbf{x},t) + \mathbf{J}_\varepsilon(\mathbf{x},t) = \mathbf{F}(\mathbf{x},t),\\
\boldsymbol{\mu}_\varepsilon(\mathbf{x})\partial_t\mathbf{H}_\varepsilon(\mathbf{x},t) + \mathbf{curl}\ \mathbf{E}_\varepsilon(\mathbf{x},t) = 0,\\
\nabla\cdot(\boldsymbol{\eta}_\varepsilon(\mathbf{x})\mathbf{E}_\varepsilon(\mathbf{x},t)) = \rho(\mathbf{x},t), \quad \nabla\cdot(\boldsymbol{\mu}_\varepsilon(\mathbf{x})\mathbf{H}_\varepsilon(\mathbf{x},t)) = 0,\\
\partial_t\mathbf{J}_\varepsilon(\mathbf{x},t) + \gamma_e\mathbf{J}_\varepsilon(\mathbf{x},t) + \omega_{e0}\displaystyle\int_0^t \mathbf{J}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon(\mathbf{x})\omega_{pe}^2\mathbf{E}_\varepsilon(\mathbf{x},t),
\end{cases}
\tag{4}
$$

where the parameters $\boldsymbol{\eta}_\varepsilon(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})$ are permittivity and permeability tensor inside Ω, respectively, ω_{pe} is the electric plasma frequency, ω_{e0} is the electric resonance frequency, γ_e is electric damping frequency, the current $\mathbf{F}(\mathbf{x},t)$ is assumed to be compactly supported in Ω, and $\mathbf{J}_\varepsilon(\mathbf{x},t)$ is the polarization current density. Here $\varepsilon>0$ denotes the relative size of a periodic microstructure of heterogeneous materials, i.e. $0<\varepsilon=l_p/L<1$, where l_p, L are respectively the sizes of a periodic cell and a domain Ω. If we assume that $L=1$, without loss of generality, then the reference periodic cell Q is defined as $Q=\{\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3): 0<\xi_i<1,\ i=1,2,3\}$ as shown in Fig. 1(b). If let $\boldsymbol{\xi}=\varepsilon^{-1}\mathbf{x}$, then we have $\boldsymbol{\eta}_\varepsilon(\mathbf{x})=\boldsymbol{\eta}(\frac{\mathbf{x}}{\varepsilon})=\boldsymbol{\eta}(\boldsymbol{\xi})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})=\boldsymbol{\mu}(\frac{\mathbf{x}}{\varepsilon})=\boldsymbol{\mu}(\boldsymbol{\xi})$. Here $\boldsymbol{\eta}_\varepsilon^{-1}(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon^{-1}(\mathbf{x})$ denote the inverse matrices of $\boldsymbol{\eta}_\varepsilon(\mathbf{x})$ and $\boldsymbol{\mu}_\varepsilon(\mathbf{x})$, respectively.

Remark 1.1. *It should be stated that the interaction of electrons or charged particles with an electric field is often treated classically by the equation of motion named the DLS model*[39] *where the polarization current satisfies the following equation:*

$$\partial_t\mathbf{J}_\varepsilon(\mathbf{x},t) + \gamma_e\mathbf{J}_\varepsilon(\mathbf{x},t) + \omega_{e0}\int_0^t \mathbf{J}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon(\mathbf{x})\omega_{pe}^2\mathbf{E}_\varepsilon(\mathbf{x},t).$$

The model is often called a Lorentz oscillator which gives rise to polarization density, and thus, polarization current. Many models follow from this model. When $\omega_{e0}=0$, the model reduces to the Drude model. Furthermore, if we set $\gamma_e=0$, the cold plasma model follows. Without loss of generality, in the rest of this article, we will assume that all of the physical parameters are positive. By using the above equation, we get

$$\mathbf{J}_\varepsilon(\mathbf{x},t) = \eta_\varepsilon\omega_{pe}^2\int_0^t g(t-\tau)\mathbf{E}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon\omega_{pe}^2 g(t)*\mathbf{E}_\varepsilon(\mathbf{x},t),$$

where $g(t)=\frac{1}{\alpha}e^{-\delta t}\sin(\alpha t)$, $\delta=\frac{\gamma_e}{2}$, $\alpha=\sqrt{\omega_{e0}^2-\delta^2}$, the symbol $*$ denotes a convolution of functions. Notice that $\omega_{e0}>\delta$ in many real applications (see, e.g., [24]*). If assume that $\omega_{e0}=0$, then we have*

$$\mathbf{J}_\varepsilon(\mathbf{x},t) = \eta_\varepsilon\omega_{pe}^2\int_0^t e^{-\gamma_e(t-\tau)}\mathbf{E}_\varepsilon(\mathbf{x},\tau)d\tau = \eta_\varepsilon\omega_{pe}^2 e^{-\gamma_e t}*\mathbf{E}_\varepsilon(\mathbf{x},t).$$

Furthermore, the transmission conditions across the boundary $\partial\Omega$ are imposed for $t>0$:

$$\mathbf{n}\times\mathbf{E}_\varepsilon = \mathbf{n}\times\mathbf{E}^{inc} + \mathbf{n}\times\mathbf{E}^{sc}, \qquad \mathbf{n}\times\mathbf{H}_\varepsilon = \mathbf{n}\times\mathbf{H}^{inc} + \mathbf{n}\times\mathbf{H}^{sc}, \tag{5}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

In addition, the initial conditions in Ω or Ω_e are given by

$$\mathbf{E}_\varepsilon(\mathbf{x},0)=\mathbf{U}_0(\mathbf{x}),\ \mathbf{H}_\varepsilon(\mathbf{x},0)=\mathbf{V}_0(\mathbf{x}),\ \mathbf{J}_\varepsilon(\mathbf{x},0)=\mathbf{0}, \tag{6}$$

where $\mathbf{U}_0(\mathbf{x})$ and $\mathbf{V}_0(\mathbf{x})$ are some given functions.