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A FINITE-DIFFERENCE SCHEME FOR A LINEAR MULTI-TERM FRACTIONAL-IN-TIME DIFFERENTIAL EQUATION WITH CONCENTRATED CAPACITIES

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Abstract. In this paper, we consider a linear multi-term subdiffusion equation with coefficients which contain Dirac distributions. Also, we consider subdiffusion equations with dynamical boundary conditions. The existence of generalized solutions of these initial-boundary value problems is proved. An implicit finite difference scheme is proposed and its stability and convergence rate are investigated in both cases. The corresponding difference schemes are tested on numerical examples.

Key words. Fractional derivative, fractional PDE, boundary value problem, interface problem, finite differences.

1. Introduction

Fractional calculus has been used as a powerful mathematical tool for the description of many phenomena in applied science. For example, fractional partial differential equations emerge in the modelling of diverse processes such as anomalous diffusion, processes in continuum mechanics as well as processes that occur in viscoelastic media, porous materials, fluids etc. ([7],[13],[14],[11]). In general, fractional derivatives are used for modeling processes with memory effects. Because of the presence of an integral in the definition of fractional derivative, it is clear that they are nonlocal operators.

The analytical solution of differential equations involving fractional derivatives, in some simple cases, can be obtained by using the Laplace transform, the Fourier transform, the Melin transform and some other techniques. Many authors have investigated numerical algorithms including finite difference methods and finite element methods ([8],[18],[12]).

In ([10]) an initial boundary value problem for a generalized multi-term fractional diffusion equation is considered. Solutions of Dirichlet and Robin boundary value problems for multi-term variable distributed order diffusion equations are studied in ([2]). In this article we consider an initial boundary value problem for a multi-term fractional in time equation with an interface. The coefficients of the equation may contain Dirac's delta distribution. It is the so-called problem with concentrated capacity.

The paper is organized as follows. In Section 2 we introduce the Riemann–Liouville and the Caputo derivatives and we mention their basic properties. In Section 3 some new function spaces are defined, especially spaces involving functions with fractional derivatives and anisotropic Sobolev spaces. In Section 4 we formulate the initial-boundary value problem for a linear multi-term fractional in time differential equation with concentrated capacities and define its weak solution. The existence and uniqueness of its (weak) solution are proved. We propose an implicit finite difference scheme and discuss its stability. The analysis of the error and

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the convergence rate of the scheme are presented in this section. One numerical example which is in agreement with the theoretical results is also presented. In Section 5 we consider a subdiffusion equation with a dynamical boundary conditions. In addition to the existence and uniqueness of the solution, a finite difference method is derived, together with its error analysis and convergence rate estimation. At the end, as in Section 4, one numerical example which is in agreement with theoretical results is also presented.

2. Fractional derivatives

Let u be a function defined on a nonempty bounded interval $[a, b]$ and let $k-1 \leq \alpha < k$, $k \in \mathbb{N}$. The left Riemann-Liouville fractional derivative of order α is defined as [14]

$$\partial_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_a^t \frac{u(s)}{(t-s)^{\alpha+1-k}}\, ds, \qquad t \geq a, \tag{1}$$

where the $\Gamma(\cdot)$ is the Gamma function. The right Riemann-Liouville fractional derivative $\partial_{b-}^{\alpha} u(t)$ is defined analogously.

The Caputo fractional derivative is obtained by interchanging the derivative and integral operators in (1)

$$^{C}\partial_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{u^{(k)}(s)}{(t-s)^{\alpha+1-k}}\, ds. \tag{2}$$

These two definitions are not equivalent and are related by the relation

$$\partial_{a+}^{\alpha} u(t) = {}^{C}\partial_{a+}^{\alpha} u(t) + \sum_{j=0}^{k-1} u^{(j)}(a) \frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)}.$$

In particular, $\partial_{a+}^{\alpha} u(t) = {}^{C}\partial_{a+}^{\alpha} u(t)$ if $u(a) = u'(a) = \cdots = u^{(k-1)}(a) = 0$.

Let us mention two properties of fractional derivatives that will be used hereafter.

For $0 < \alpha < 1$ and continuously differentiable functions $u(t)$ and $v(t)$, the following equality holds:

$$(\partial_{a+}^{\alpha} u,\, v)_{L^2(a,b)} = (u,\, \partial_{b-}^{\alpha} v)_{L^2(a,b)}\,. \tag{3}$$

Also, if $\alpha > 0$ and if u is an infinitely differentiable function in $\mathbb{R}$, with $\operatorname{supp} u \subset (a, b)$, then u satisfies the following relation (see [5]):

$$(\partial_{a+}^{\alpha} u,\, \partial_{b-}^{\alpha} u)_{L^2(a,b)} = \cos \pi\alpha \, \|\partial_{a+}^{\alpha} u\|_{L^2(a,+\infty)}^2\,. \tag{4}$$

For functions of several variables, partial fractional derivatives are defined in an analogous manner, for example,

$$\partial_{t,a+}^{\alpha} u(x,t) = \frac{1}{\Gamma(k-\alpha)} \frac{\partial^k}{\partial t^k} \int_a^t \frac{u(x,s)}{(t-s)^{\alpha+1-k}}\, ds, \quad k-1 < \alpha < k, \quad k \in \mathbb{N}.$$

3. Some function spaces

First, we introduce some notations and define some function spaces, norms and inner products that are used hereafter. Let Ω be an open domain in $\mathbb{R}^n$. As usual, by $C^k(\Omega)$ and $C^k(\overline{\Omega})$ we denote the spaces of k-fold differentiable functions defined on Ω. By $\dot{C}^{\infty}(\Omega) = C_0^{\infty}(\Omega)$ we denote the space of infinitely differentiable functions with compact support in Ω. The space of measurable functions whose