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ERROR ESTIMATES FOR THE LAPLACE INTERPOLATION ON CONVEX POLYGONS

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Abstract. In the natural element method (NEM), the Laplace interpolation error estimate on convex planar polygons is proved in this study. The proof is based on bounding gradients of the Laplace interpolation for convex polygons which satisfy certain geometric requirements, and has been divided into several parts that each part is bounded by a constant. Under the given geometric assumptions, the optimal convergence estimate is obtained. This work provides the mathematical analysis theory of the NEM. Some numerical examples are selected to verify our theoretical result.

Key words. Natural element method, geometric constraints, Laplace interpolation, error estimate.

1. Introduction

In engineering practice, many problems in mechanics and physics can be reduced to solve mathematical problems of ordinary differential equations or partial differential equations under given boundary conditions. With the rapid development of computer technology, many numerical methods such as weighted residual method, finite element method, finite difference method, meshless method [1] and boundary element method have been developed to solve the engineering problems.

In recent decades, meshless methods have emerged, such as smooth particle hydrodymics method (SPH) [2, 3], reproducing kernel particle method (RKPM) [4], moving least-square approximation method (MLS) [5], the partition of unity method [6], radial basis functions (RBF) [7], point interpolation method (PIM) [8] and natural element method (NEM) [9], and these methods have been developed to solve partial differential equations (PDEs). In the meshless method, shape functions are constructed in terms of a group of discrete nodes, and no predefined nodal connectivity is required. The nodes are unstructured and can be freely moved, inserted and deleted. The meshless method does not need to generate mesh, thus it has some advantages in handling crack propagation or large deformation problems.

The NEM is a meshless method based on the concept of natural neighbor interpolants and on the discretization by Voronoi diagram and Delaunay triangulation. The NEM interpolant is constructed on the basis of Voronoi diagram [10], which is unique for a given set of distinct nodes in the plane. It means that the NEM interpolant is determined once the location of nodes is determined. The dual Delaunay triangles [11] are constructed for nodal integration and numerical computation of the interpolant. However, unlike finite element method (FEM) where angle restrictions are imposed on the triangles for the convergence of the method, there are no such constraints on the size, shape, and angles of the triangles in NEM. Unlike most of meshless methods, the natural neighbor interpolants have the properties of interpolation of nodal data, allowing direct imposition of essential boundary conditions as FEM does. On the other hand, the NEM presents some characteristics of meshless methods, such as accurate shape functions with quasi-spherical influence

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zones and robust approximations with no user-defined parameter on non-uniform grids, and it can handle complex geometry or crack propagation problems easily. In general, the NEM not only has the advantages of finite element method and meshless method, but also overcomes some of their shortcomings.

Since Braun and Sambridge [9] firstly introduced NEM in 1995, many researchers have applied it to solve mechanical problems. Sukumar et al. [12, 13] used NEM to study the application of elliptic boundary value problems for elastic mechanics, and constructed the C^1 natural neighbor interpolation to solve fourth-order elliptic partial differential equations. Cueto et al. [14, 15] used α-shapes in the context of NEM to ensure the linear precision of the interpolant over convex and non-convex boundaries. Bueche et al. [16] investigated NEM in two-dimensional linear elastodynamics and studied vibration and wave propagation problem. Toi Yutaka [17, 18] analyzed the elastic-plastic problem and brittle fracture problem with NEM. Cai and Zhu [19, 20] used a local Petrov-Galerkin method to establish a global equilibrium equation, which has fast convergence rate and high accuracy. Gonzalez [21] established a novel algorithm to simulate free-surface fluid dynamics phenomena. Alfaro [22] used NEM to simulate hollow profiles. Cho et al. [23] presented a mixed natural element approximation of Reissner-Mindlin plate for the locking-free numerical analysis of plate-like thin elastic structures.

The application of NEM has been developed for about twenty years [24, 25], while the theoretical research of convergence, stability and error analysis is rare and needs to be studied deeply. Gillette et al. [26] made a brief study on the error estimates for generalized barycentric interpolation, including the Sibson interpolation. Alexander et al. [27] proved interpolation error estimates for the mean value coordinates over convex polygons. In a similar fashion to estimates shown for different coordinates in these papers, we study another interpolation error of NEM-Laplace interpolation.

The rest of the paper is organized as follows. In Section 2, we review the relevant background on geometric constraints, Laplace interpolation and interpolation theory in Sobolev Spaces. In Section 3, the estimate is divided into several parts and the initial estimate is established for each part. Our result is obtained in Theorem 2 which gives a constant bound on the gradients of the Laplace interpolation. In Section 4, two numerical examples are given to verify our analysis. Finally the conclusion is drawn in Section 5.

2. Background

2.1. Geometric constraints. Let Ω be a convex polygon in $\mathbb{R}^2$, which consists of n nodes, $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$, and let the interior angle at $\boldsymbol{x}_i$ be β_i. The largest distance between two points in Ω is denoted by $diam(\Omega)$ and the radius of the largest inscribed circle is denoted $\rho(\Omega)$, then the aspect ratio γ is defined as $\gamma := \frac{diam(\Omega)}{\rho(\Omega)}$. Now we give the following geometric constraints.

G1. Bounded aspect ratio: There exists a constant $\gamma^* > 0$ such that $2 \leq \gamma \leq \gamma^*$.

G2. Minimum edge length: There exists a constant $d^* > 0$ such that $|\boldsymbol{x}_i - \boldsymbol{x}_j| \geq d^*$ for all $i \neq j$.

G3. Maximum interior angle: There exists a constant $\beta^* > 0$ such that $\beta_i < \beta^* < \pi$ for all i.

Under the above geometric constraints, two other closely related properties also hold.

G4. Minimum interior angle: There exists a constant $\beta_* > 0$ such that $\beta_i > \beta_* > 0$ for all i.