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STABILITY OF HIGH ORDER FINITE DIFFERENCE SCHEMES WITH IMPLICIT-EXPLICIT TIME-MARCHING FOR CONVECTION-DIFFUSION AND CONVECTION-DISPERSION EQUATIONS

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Abstract. The main purpose of this paper is to analyze the stability of the implicit-explicit (IMEX) time-marching methods coupled with high order finite difference spatial discretization for solving the linear convection-diffusion and convection-dispersion equations in one dimension. Both Runge-Kutta and multistep IMEX methods are considered. Stability analysis is performed on the above mentioned schemes with uniform meshes and periodic boundary condition by the aid of the Fourier method. For the convection-diffusion equations, the result shows that the high order IMEX finite difference schemes are subject to the time step restriction $\Delta t \leq \max\{\tau_0, c\Delta x\}$, where τ_0 is a positive constant proportional to the diffusion coefficient and c is the Courant number. For the convection-dispersion equations, we show that the IMEX finite difference schemes are stable under the standard CFL condition $\Delta t \leq c\Delta x$. Numerical experiments are also given to verify the main results.

Key words. Convection-diffusion equation, convection-dispersion equation, stability, IMEX, finite difference, Fourier method.

1. Introduction

In this paper, the stability property of the high order finite difference schemes with certain implicit-explicit (IMEX) time-marching methods is studied for the convection-diffusion and convection-dispersion equations respectively. For the spatial derivative terms of these equations, we use a high order upwind biased finite difference scheme, which is a prototype of the weighted essentially non-oscillatory (WENO) schemes [12, 14], to discretize the convection term, a high order central difference method to discretize the diffusion term, and a high order upwind biased finite difference scheme to discretize the dispersion term.

The time derivative term for the convection-diffusion and convection-dispersion equations should be discretized carefully. If explicit time-marching methods are used, then the time step is dominated by the highest order derivative term, which may be very small, resulting in excessive computational cost. For example, for the convection-dispersion equations involving third order spatial derivatives which are not convection-dominated, the explicit time discretization may suffer from a strict time step restriction $\Delta t \sim O(\Delta x^3)$ for stability, where Δt is the time step and Δx is the spatial mesh size. If the fully implicit time-marching methods are used, then the time step restriction may be relaxed, and usually unconditionally stable such as A-stable schemes can be designed. However, in many practical applications the lower order convection terms are often nonlinear, hence the implicit methods may be much more expensive per time step than the explicit methods, because an iterative solution of the nonlinear algebraic equations is needed.

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When it comes to such problems, a natural consideration is to treat different derivative terms differently, that is, the higher order derivative terms are treated implicitly, whereas the rest of the terms are treated explicitly. The IMEX time-marching methods, which have been proposed and studied by many authors [1–7, 9, 10, 13, 16, 17], have considered such a strategy. This can not only alleviate the stringent time step restriction, but also reduce the difficulty of solving the algebraic equations, especially when the higher order derivative terms are linear. Even when the higher order derivative terms are nonlinear, the IMEX time-marching methods might still show their advantages in obtaining a better algebraic system, for example for diffusive higher order derivatives the algebraic system might have some symmetry and positive definite properties, which can be easily solved by many iterative methods.

For the convection-diffusion equations, there have been many studies in the literature on the IMEX methods. In [1], a pair of multistep IMEX time-marching methods are constructed. Coupled with the traditional second order central difference method, the multistep IMEX finite difference schemes are shown to be stable under the standard CFL condition $\Delta t \le c\Delta x$, where c is the Courant number. However, most of them tend to have an undesirably small c, unless diffusion strongly dominates and an appropriate backward differentiation formula is selected for the diffusion term. In [9], the authors designed several stable multistep IMEX time discretizations, which are specially tailored for stability when coupled with the pseudospectral method. These schemes are shown to be stable provided that the time step and the spatial mesh size are bounded by two constants. Combined with the local discontinuous Galerkin (LDG) method, a variety of IMEX schemes [16, 17], including Runge-Kutta type and multistep type IMEX schemes, have been discussed. These schemes are stable provided that the time step is upper-bounded by a positive constant τ_0 which is proportional to d/ν^2, where ν and d are the convection and diffusion coefficients, respectively. However, when d is very small in comparison with the spatial mesh size, τ_0 is too small to be the true bound for stability. For the above mentioned equations without the diffusion terms, the explicit scheme is usually stable under the standard CFL condition. We could therefore reasonably expect that the IMEX method for this convection-diffusion equation should also be stable under the same CFL condition. The schemes in [18], where the explicit part is treated by a strong-stability-preserving Runge-Kutta method [8], and the implicit part is treated by an L-stable diagonally implicit Runge-Kutta method, are also subject to the time step restriction $\Delta t \le \tau_0$. They also face the problem that τ_0 is too small to be the true bound for stability when d is very small in comparison with the spatial mesh size.

For the convection-dispersion equations, there are also some studies in the literature on the IMEX methods. In [6], some multistep IMEX time-marching methods with the spectral spatial discretization for the KdV equation have been presented. Coupled with the finite volume spatial discretization, some IMEX Runge-Kutta methods are tested in the case of the KdV equation in [5]. These schemes are shown to be stable under the standard CFL condition $\Delta t \le c\Delta x$. In [10], the IMEX method with the discontinuous Galerkin (DG) spatial discretization is proposed for the KdV equation, where the stability analysis is not discussed.

If we summarize the stability conditions of the schemes mentioned above, we could find that the explicit, implicit and IMEX schemes coupled with appropriate spatial discretizations are subject to the time step restrictions shown in Table 1. Notice that the specific choices of spatial discretizations may change the values of