INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 4, Pages 481–499

AN *hp* FINITE ELEMENT METHOD FOR A SINGULARLY PERTURBED REACTION-CONVECTION-DIFFUSION BOUNDARY VALUE PROBLEM WITH TWO SMALL PARAMETERS

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This paper is dedicated to Manil Suri on the occasion of his 60^{th} *birthday*

Abstract. We consider a second order singularly perturbed boundary value problem, of reaction-convection-diffusion type with two small parameters, and the approximation of its solution by the *hp* version of the Finite Element Method on the so-called *Spectral Boundary Layer* mesh. We show that the method converges uniformly, with respect to both singular perturbation parameters, at an exponential rate when the error is measured in the energy norm. Numerical examples are also presented, which illustrate our theoretical findings as well as compare the proposed method with others found in the literature.

Key words. Singularly perturbed problem, reaction-convection-diffusion, boundary layers, *hp* finite element method, robust exponential convergence.

1. Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last few decades (see, e.g., the books [15], [16], [20] and the references therein). As is well known, a main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall reliability of the approximate solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform, layer-adapted meshes (such as the Shishkin [24] or Bakhvalov [2] mesh), or the use of the high order p and hp versions on the so-called *Spectral Boundary Layer* mesh [11], [23].

Usually, problems of convection-diffusion or reaction-diffusion type are studied separately and several researchers have proposed and analyzed numerical schemes for the robust approximation of their solution (see, e.g., [20] and the references therein). When there are two singular perturbation parameters present in the differential equation, the problem becomes reaction-convection-diffusion and the relationship between the parameters determines the 'regime' we are in (see Table 1 ahead). In [6] this problem was addressed using the h version of the FEM as well as appropriate finite differences (see also [3], [5], [7], [17], [21], [28], [29]). In the present article we consider the hp version of the FEM on the *Spectral Boundary Layer* mesh (from [11]) and show that the method converges uniformly in the perturbation parameters at an exponential rate, when the error is measured in the energy norm.

Received by the editors October 22, 2019 and, in revised form, April 16, 2021.
2000 *Mathematics Subject Classification.* 65N30.

The rest of the paper is organized as follows: in Section 2 we present the model problem and its regularity. Section 3 presents the discretization using the *Spectral Boundary Layer* mesh and contains our main result of uniform, exponential convergence. Finally, in Section 4 we show the results of numerical computations that illustrate and extend our theoretical findings.

With $I \subset \mathbb{R}$ an interval with boundary ∂I and measure $|I|$, we will denote by $C^k(I)$ the space of continuous functions on I with continuous derivatives up to order k. We will use the usual Sobolev spaces $W^{k,m}(I)$ of functions on I with $0, 1, 2, ..., k$ generalized derivatives in $L^m(I)$, equipped with the norm and seminorm $\|\cdot\|_{k,m,I}$ and $|\cdot|_{k,m,I}$, respectively. When $m = 2$, we will write $H^k(I)$ instead of $W^{k,2}(I)$, and for the norm and seminorm, we will write $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}$, respectively. The usual $L^2(I)$ inner product will be denoted by $\langle \cdot, \cdot \rangle_I$, with the subscript omitted when there is no confusion. We will also use the space

$$H_0^1(I) = \left\{ u \in H^1(I) : u|_{\partial I} = 0 \right\}.$$

The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty,I}$. Finally, the notation "$a \lesssim b$" means "$a \le Cb$" with C being a generic positive constant, independent of any parameters (e.g. discretization, singular perturbation, etc.).

2. The model problem and its regularity

We consider the following model problem (cf. [14]): Find u such that

$$-\varepsilon_1 u''(x) + \varepsilon_2 b(x) u'(x) + c(x) u(x) = f(x),\ x \in I = (0,1)\,, \tag{1}$$

$$u(0) = u(1) = 0\,, \tag{2}$$

where $0 < \varepsilon_1, \varepsilon_2 \le 1$ are given parameters that can approach zero and the functions b, c, f are given and sufficiently smooth. In particular, we assume that they are analytic functions satisfying, for some positive constants $\gamma_f, \gamma_c, \gamma_b$, independent of $\varepsilon_1, \varepsilon_2$,

$$\left\| f^{(n)} \right\|_{\infty,I} \lesssim n!\gamma_f^n\ ,\ \left\| c^{(n)} \right\|_{\infty,I} \lesssim n!\gamma_c^n\ ,\ \left\| b^{(n)} \right\|_{\infty,I} \lesssim n!\gamma_b^n\ \forall\ n = 0, 1, 2, ... \tag{3}$$

In addition, we assume that there exist positive constants β, γ, ρ, independent of $\varepsilon_1, \varepsilon_2$, such that $\forall\ x \in \overline{I}$

$$b(x) \ge \beta > 0\ ,\ c(x) \ge \gamma > 0\ ,\ c(x) - \frac{\varepsilon_2}{2} b'(x) \ge \rho > 0. \tag{4}$$

The solution to (1), (2) satisfies (see, e.g., [6])

$$\|u\|_{\infty,I} \lesssim 1. \tag{5}$$

Moreover, the following result was shown in [27].

Proposition 1. *Let u be the solution of (1), (2). Then, there exists a positive constant K, independent of $\varepsilon_1, \varepsilon_2$, such that for $n = 0, 1, 2, ...$*

$$\left\| u^{(n)} \right\|_{\infty,I} \lesssim K^n \max\left\{ n, \varepsilon_1^{-1}, \varepsilon_2^{-1} \right\}^n.$$

More details arise if one studies the structure of the solution to (1), which depends on the roots of the characteristic equation associated with the differential