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GLOBAL WELL-POSEDNESS FOR NAVIER-STOKES-DARCY EQUATIONS WITH THE FREE INTERFACE

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Abstract. In this paper, the Navier-Stokes-Darcy equations with the free interface are considered, which model the movement of the sea and the sand in the seafloor or the filtration of blood through the arterial wall. The global well-posedness of the solution perturbed around the constant steady state is obtained and then the almost exponential decay to the constant stationary state is gained. Finally, we present an efficient explicit discrete scheme based on finite-volume method for the free interface system and provide the numerical tests to illustrate the consistency with our analysis result.

Key words. Navier-Stokes-Darcy equations, the free interface, global well-posedness, large time behavior.

1. Introduction

In this paper, we consider the Navier-Stokes-Darcy equations with the free boundary, that is, the viscous, incompressible fluid coupled with the porous medium flow which are separated by the free interface. The equations model the movement of the sea and the sand in the seafloor or the filtration of blood through the arterial wall; refer to [29, 2] and the references therein. The mixed Stokes-Darcy model, which is the simplified model of the Navier-Stokes-Darcy equations, has a wide range of applications in science and engineering including industrial settings, especially in cases where a free flowing fluid moves over a porous medium, referring to [8, 14, 24, 26] and the reference therein.

Let us assume that the two-phase flows are confined in a domain $\Omega \subset \mathbb{R}^3$, which is separated into two free moving regions $\Omega_+(t)$ and $\Omega_-(t)$ such that $\bar{\Omega} = \bar{\Omega}_+(t) \cup \bar{\Omega}_-(t)$ and $\Omega_+(t) \cap \Omega_-(t) = \emptyset$. Here $\Omega_+(t)$ and $\Omega_-(t)$ represent the region of the upper flow and the porous matrix region, respectively, defined as

$$\Omega_+(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid \eta(t, y') < y_3 < 1\} \tag{1}$$

and

$$\Omega_-(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid -b(y') < y_3 < \eta(t, y')\}.$$

where $y = (y', y_3)$ and $y' = (y_1, y_2)$, $\mathbb{T}^2 = (2\pi L_1 \mathbb{T}) \times (2\pi L_2 \mathbb{T})$. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the usual $1-$torus and $L_1, L_2 > 0$ are fixed constants. The interface separating the domain Ω is denoted by

$$\Sigma_-(t) = \{(y', y_3) | y_3 = \eta(t, y')\}.$$

Let

$$\Sigma_+ = \{(y', y_3) | y_3 = 1\} \quad \text{and} \quad \Sigma_{-b} = \{(y', y_3) | y_3 = -b(y')\}$$

denote the fixed upper boundary of $\Omega_+(t)$ and the given lower boundary of $\Omega_-(t)$ respectively, where $b(y') \in C^\infty(\mathbb{T}^2)$ is the known function describing the location of

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the bottom Σ_{-b}. The upper fluid is described by the incompressible Navier-Stokes equations

$$\begin{cases} \rho_w \partial_t \tilde{u} + \rho_w \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = \mu \Delta \tilde{u} - g \rho_w \vec{e}_3 \\ \operatorname{div} \tilde{u} = 0 \end{cases} \quad \text{in } \Omega_+(t), \tag{2}$$

where the vector $\tilde{u}$ and the scalar function $\tilde{p}$ denote the velocity and pressure of the fluid respectively. $\rho_w > 0$ and $\mu > 0$ denote the density and viscosity of the upper fluid, $g > 0$ is the gravitational constant and $\vec{e}_3 = (0, 0, 1)^T$ is the $Z-$axis unit vector.

The lower fluid is described by the porous medium model with $\gamma > 1$

$$\partial_t \tilde{\phi} - \Delta \tilde{\phi}^\gamma = 0, \tag{3}$$

where the scalar function $\tilde{\phi}(t, \cdot) : \Omega_-(t) \to \mathbb{R}$ denotes the hydraulic head or dynamic pressure. By the Darcy's law, the velocity of the lower fluid is defined by

$$u_- := -\frac{\gamma}{\gamma - 1} \nabla \tilde{\phi}^{\gamma - 1} \quad \text{for} \quad \gamma > 1.$$

In the present paper, we consider the following boundary condition. On the upper fixed boundary Σ_+ the non-slip condition is considered

$$\tilde{u} = 0 \qquad \text{on } \Sigma_+, \tag{4}$$

and we give some fixed constant pressure $\tilde{\phi}_b > 0$ on the lower fixed boundary Σ_{-b}, i.e.

$$\tilde{\phi} = \tilde{\phi}_b \qquad \text{on } \Sigma_{-b}. \tag{5}$$

It's an open problem that what conditions on the free interface $\Sigma_-(t)$ make the problem (2) and (3) well-posed. In the present paper, the Beavers-Joesph-Saffman's interface condition on $\Sigma_-(t)$, seeing [7, 30] for the detail, is considered

$$\begin{cases} \tilde{u} \cdot \vec{n} = u_- \cdot \vec{n} \\ (\tilde{p} I - \mu \mathbb{D}(\tilde{u})) \vec{n} = \tilde{P}_\gamma \vec{n} - \rho_s g \eta \vec{n} \end{cases} \quad \text{on } \Sigma_-(t), \tag{6}$$

where

$$\tilde{P}_\gamma = \frac{\gamma}{\gamma - 1} \tilde{\phi}^{\gamma - 1} \quad \text{for } \gamma > 1,$$

I denotes the 3×3 identity matrix, $(\mathbb{D}(\tilde{u}))_{ij} = \partial_j \tilde{u}^i + \partial_i \tilde{u}^j$ describes twice of the velocity deformation tensor, the positive constant ρ_s is the density of lower fluid, satisfying

$$\rho_s > \rho_w,$$

$\vec{n}$ is the unit normal vector of $\Sigma_-(t)$ pointing to the upper fluid, given by

$$\vec{n} = \frac{(-\partial_{y_1} \eta, -\partial_{y_2} \eta, 1)}{\sqrt{1 + |\nabla_H \eta|^2}} = \frac{\vec{N}}{\sqrt{1 + |\nabla_H \eta|^2}},$$

∇_H, div_H and Δ_H denote the horizontal gradient, the horizontal divergence and the horizontal Laplace operator respectively. According to the kinematic boundary condition of the fluid, the free interface satisfies

$$\partial_t \eta + u_{-,1} \partial_{y_1} \eta + u_{-,2} \partial_{y_2} \eta = u_{-,3} \qquad \text{on } \Sigma_-(t), \tag{7}$$

where $u_{-,i} (i = 1, 2, 3)$ represents the i-th element of the velocity u_-. The initial data are given by

$$(\tilde{u}, \tilde{\phi}, \eta)|_{t=0} = (\tilde{u}_0, \tilde{\phi}_0, \eta_0), \tag{8}$$