INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 5, Pages 620–641

CONVERGENCE ANALYSIS OF ADI ORTHOGONAL SPLINE COLLOCATION WITHOUT PERTURBATION TERMS

BERNARD BIALECKI AND RYAN I. FERNANDES

Abstract. For the heat equation on a rectangle and nonzero Dirichlet boundary conditions, we consider an ADI orthogonal spline collocation method without perturbation terms, to specify boundary values of intermediate solutions at half time levels on the vertical sides of the rectangle. We show that, at each time level, the method has optimal convergence rate in the L^2 norm in space. Numerical results for splines of orders $4, 5, 6$ confirm our theoretical convergence rates and demonstrate suboptimal convergence rates in the H^1 norm. We also demonstrate numerically that the scheme without the perturbation terms is applicable to variable coefficient problems yielding the same convergence rates obtained for the heat equation.

Key words. Convergence, alternating direction implicit method, orthogonal spline collocation, perturbation terms.

1. Introduction

The alternating direction implicit (ADI) method is a popular and useful technique for solving partial differential equations on rectangles. Such methods reduce the solution of multi-dimensional problems to the solution of a collection of independent discrete one-dimensional problems in the coordinate directions. ADI techniques have been used in recent years to solve a variety of problems in various fields such as biology, engineering, finance, physics (see, for example, [1, 8, 13, 14, 16, 19, 23, 25, 31, 32]).

ADI methods were first introduced, in the context of finite differences, by Peaceman and Rachford [20] to solve parabolic and elliptic problems with zero Dirichlet boundary conditions. When extending the ADI finite difference method to nonzero Dirichlet boundary conditions, some authors included additional terms, called 'perturbation terms', to specify intermediate solutions at half time levels on vertical sides of the rectangle (see, for example, [12, (2.8)], [27, (13), (14) on pg. 549, (35) on pg. 555], [29, (7.3.11)], [30, (4.4.20), (4.4.21)]). The inclusion of perturbation terms preserves the optimal convergence rate in the discrete H^1 norm in space. However, it has been shown in [17, 3] for the heat equation and a variable coefficient parabolic equation, respectively, that the ADI finite difference scheme without perturbation terms has optimal convergence rate in the discrete L^2 norm in space. This important finding opened the door to an application of the ADI finite difference method to the solution of parabolic equations with Dirichlet boundary conditions on nonrectangular sets. In [4], for the first time in the literature, we have formulated and analyzed an ADI finite difference method without the perturbation terms on a convex set.

Over the past several years ADI orthogonal spline collocation (OSC) has proved to be an efficient technique to solve time dependent partial differential equation problems on rectangles and rectangular polygons (see [5, 6, 13, 14, 15, 16, 22, 24, 26] and references therein). The ADI OSC scheme was analyzed in [15] for the solution of the heat equation with zero Dirichlet boundary conditions on a rectangle. The

Received by the editors March 18, 2021 and, in revised form, May 18, 2021.
2000 *Mathematics Subject Classification.* 65M70, 65M12.

ADI OSC scheme with perturbation terms was analyzed in [5] for the solution of a variable coefficient parabolic equation with nonzero Dirichlet boundary conditions on a rectangle. The purpose of the present paper is to prove the optimal convergence rate in the L^2 norm of the ADI OSC scheme without perturbation terms for the solution of the heat equation

$$u_t + (L_1 + L_2)u = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \tag{1}$$

where $\Omega = (a, b) \times (c, d)$,

$$L_1 u = -u_{xx}, \quad L_2 u = -u_{yy}, \tag{2}$$

with the initial and nonzero Dirichlet boundary conditions given by

$$u(x, y, 0) = g_1(x, y), \quad (x, y) \in \overline{\Omega}, \tag{3}$$

$$u(x, y, t) = g_2(x, y, t), \quad (x, y, t) \in \partial\Omega \times (0, T]. \tag{4}$$

While we define the ADI OSC scheme and give convergence analysis for the heat equation, we demonstrate by a numerical example that the scheme without the perturbation terms is applicable to variable coefficient parabolic problems yielding the same convergence rates as those for the heat equation. We expect the result of this paper to impact applications and convergence analysis of the ADI OSC method for parabolic equations with nonzero Dirichlet boundary conditions on non-rectangular sets [7].

In section 2 we give Preliminaries. The ADI OSC schemes with and without perturbation terms are described in section 3. Convergence analysis of the ADI scheme without perturbation terms is carried out in section 4. In section 5, errors and convergence rates of the ADI OSC schemes with and without perturbation terms are presented for splines of orders 4, 5, 6. Concluding remarks are given in section 6.

2. Preliminaries

Let $\{x_i\}_{i=0}^{N_x}$ and $\{y_j\}_{j=0}^{N_y}$ be respectively partitions (in general nonuniform) of $[a, b]$ and $[c, d]$ such that

$$a = x_0 < x_1 < \cdots < x_{N_x-1} < x_{N_x} = b, \qquad c = y_0 < y_1 < \cdots < y_{N_y-1} < y_{N_y} = d.$$

Let $I_i^x = (x_{i-1}, x_i)$, $I_j^y = (y_{j-1}, y_j)$, $h_i^x = x_i - x_{i-1}$, $h_j^y = y_j - y_{j-1}$, and let

$$\underline{h}_x = \min_i h_i^x, \quad \overline{h}_x = \max_i h_i^x, \quad \underline{h}_y = \min_j h_j^y, \quad \overline{h}_y = \max_j h_j^y,$$

$$h = \max(\overline{h}_x, \overline{h}_y).$$

We assume that a collection of the partitions $\{x_i\}_{i=0}^{N_x} \times \{y_j\}_{j=0}^{N_y}$ of Ω is regular, that is, there exist positive constants σ_1, σ_2, and σ_3 such that for every partition in the collection, we have

$$\sigma_1 \overline{h}_x \le \underline{h}_x, \quad \sigma_1 \overline{h}_y \le \underline{h}_y, \quad \sigma_2 \le \frac{\overline{h}_x}{\overline{h}_y} \le \sigma_3.$$

In the following, we assume that a natural number $r \ge 3$. Let P_r denote the set of polynomials of degree $\le r$. Let $\mathcal{M}_x$, $\mathcal{M}_x^0$, $\mathcal{M}_y$, and $\mathcal{M}_y^0$ be the spaces defined by

$$\mathcal{M}_x = \{v \in C^1[a, b] : v|_{[x_{i-1}, x_i]} \in P_r, i = 1, \ldots, N_x\},$$

$$\mathcal{M}_x^0 = \{v \in \mathcal{M}_x : v(a) = v(b) = 0\},$$

$$\mathcal{M}_y = \{v \in C^1[c, d] : v|_{[y_{j-1}, y_j]} \in P_r, j = 1, \ldots, N_y\},$$

$$\mathcal{M}_y^0 = \{v \in \mathcal{M}_y : v(c) = v(d) = 0\}.$$