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MODIFIED TIKHONOV REGULARIZATION FOR IDENTIFYING SEVERAL SOURCES

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Abstract. We study whether a modified version of Tikhonov regularization can be used to identify several local sources from Dirichlet boundary data for a prototypical elliptic PDE. This paper extends the results presented in [5]. It turns out that the possibility of distinguishing between two, or more, sources depends on the smoothing properties of a second or fourth order PDE. Consequently, the geometry of the involved domain, as well as the position of the sources relative to the boundary of this domain, determines the identifiability. We also present a uniqueness result for the identification of a single local source. This result is derived in terms of an abstract operator framework and is therefore not only applicable to the model problem studied in this paper. Our schemes yield quadratic optimization problems and can thus be solved with standard software tools. In addition to a theoretical investigation, this paper also contains several numerical experiments.

Key words. Inverse source problems, PDE-constrained optimization, Tikhonov regularization, nullspace, numerical computations.

1. Introduction

We will study the following problem:

$$\min_{(f,u)\in F_h\times H^1(\Omega)} \left\{ \frac{1}{2}\|u-d\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\alpha\|\mathsf{W}f\|^2_{L^2(\Omega)} \right\} \tag{1}$$

subject to

$$\begin{aligned} -\Delta u + \epsilon u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where F_h is a finite dimensional subspace of $L^2(\Omega)$, $\mathsf{W}: F_h \to F_h$ is a linear regularization operator, α is a regularization parameter, d represents Dirichlet boundary data, ϵ is a positive constant, $\mathbf{n}$ denotes the outwards pointing unit normal vector of the boundary $\partial\Omega$ of the bounded domain Ω, and f is the source. Depending on the choice of W, we obtain different regularization terms, including the standard version $\mathsf{W} = I$ (the identity map).

The purpose of solving (1)-(2) is to estimate the unknown source f from the Dirichlet boundary data $u = d$ on $\partial\Omega$. Mathematical problems similar to this occur in numerous applications, e.g., in connection with electroencephalography (EEG) and electrocardiography (ECG), and has been studied by many scientists, see, e.g., [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. A more detailed description of previous investigations is presented in [5].

In [5] we showed with mathematical rigor that a particular choice of W *almost* enables the identification of the position of a *single* local source from the boundary data. That paper also contains numerical experiments suggesting that two or three local sources, in some cases, can be recovered. The purpose of this paper is to explore the *several sources* situation in more detail, both theoretically and

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experimentally. Moreover, we prove that our particular choice of W, which will be presented below, enables the *precise* recovery of a single local source.

2. Analysis

2.1. Results for general problems. Let us consider the abstract operator equation

$$\mathsf{K_h}\mathbf{x} = \mathbf{b}, \tag{3}$$

where $\mathsf{K_h} : X \to Y$ is a linear operator with a *nontrivial nullspace* and possibly very small singular values, X and Y are real Hilbert spaces, X is finite dimensional and $\mathbf{b} \in Y$. We will use the notation $\|\cdot\|_X$ and $\|\cdot\|_Y$ for the norms induced by the inner products of X and Y, respectively. (For the problem (1)-(2), $\mathsf{K_h}$ is the forward operator

$$\mathsf{K_h} : F_h \to L^2(\partial\Omega), \quad f \mapsto u|_{\partial\Omega},$$

where F_h is a finite dimensional subspace of $L^2(\Omega)$, and u is the unique solution of the boundary value problem (2) for a given f.)

Applying traditional Tikhonov regularization, with the regularization parameter $\alpha > 0$, yields the approximation

$$\mathbf{x}_\alpha = \arg\min_{\mathbf{x}} \left\{ \frac{1}{2}\|\mathsf{K_h}\mathbf{x} - \mathbf{b}\|_Y^2 + \frac{1}{2}\alpha\|\mathbf{x}\|_X^2 \right\}, \tag{4}$$

and, according to standard theory, the minimum norm least squares solution $\mathbf{x}^*$ of (3) satisfies

$$\mathbf{x}^* = \lim_{\alpha \to 0} \mathbf{x}_\alpha = \mathsf{K_h}^\dagger \mathbf{b} \in \mathcal{N}(\mathsf{K_h})^\perp,$$

where $\mathcal{N}(\mathsf{K_h})^\perp$ denotes the orthogonal complement of the nullspace $\mathcal{N}(\mathsf{K_h})$ of $\mathsf{K_h}$, and $\mathsf{K_h}^\dagger$ represents the Moore-Penrose inverse of $\mathsf{K_h}$.

Throughout this paper we assume that

$$\mathcal{B} = \{\phi_1, \phi_2, \ldots, \phi_n\}$$

is an *orthonormal basis* for X and that

$$\mathsf{K_h}(\phi_i) \neq c\mathsf{K_h}(\phi_j) \quad \text{for } i \neq j \text{ and } c \in \mathbb{R}. \tag{5}$$

That is, the images under $\mathsf{K_h}$ of the basis functions are not allowed to be parallel. Note that (5) asserts that none of the basis functions belong to the nullspace $\mathcal{N}(\mathsf{K_h})$ of $\mathsf{K_h}$. (For PDE-constrained optimization problems one can, e.g., choose basis functions with local support. We will return to this matter in subsection 2.2.)

Throughout this text,

$$\mathsf{P} : X \to \mathcal{N}(\mathsf{K_h})^\perp \tag{6}$$

denotes the orthogonal projection of elements in X onto $\mathcal{N}(\mathsf{K_h})^\perp$. In [5] we investigated whether a single basis function ϕ_j can be recovered from its image[1] $\mathsf{K_h}\phi_j$. More specifically, using the fact that $\mathsf{K_h}^\dagger\mathsf{K_h} = \mathsf{P}$, we observe that the minimum norm least squares solution $\mathbf{x}_j^*$ of

$$\mathsf{K_h}\mathbf{x} = \mathsf{K_h}\phi_j \tag{7}$$

is

$$\mathbf{x}_j^* = \mathsf{K_h}^\dagger(\mathsf{K_h}\phi_j) = \mathsf{P}\phi_j. \tag{8}$$

[1]Since $\mathsf{K_h}$ has a nontrivial nullspace, it is by no means obvious that ϕ_j can be recovered from its image $\mathsf{K_h}(\phi_j)$.