

STABILITY ANALYSIS AND ERROR ESTIMATES OF LOCAL DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION EQUATIONS ON OVERLAPPING MESH WITH NON-PERIODIC BOUNDARY CONDITIONS

NATTAPORN CHUENJARERN, KANOGNUDGE WUTTANACHAMSRI, AND YANG
YANG

Abstract. A new local discontinuous Galerkin (LDG) method for convection-diffusion equations on overlapping meshes with periodic boundary conditions was introduced in [14]. With the new method, the primary variable u and the auxiliary variable $p = u_x$ are solved on different meshes. In this paper, we will extend the idea to convection-diffusion equations with non-periodic boundary conditions, i.e. Neumann and Dirichlet boundary conditions. The main difference is to adjust the boundary cells. Moreover, we study the stability and suboptimal error estimates. Finally, numerical experiments are given to verify the theoretical findings.

Key words. Local discontinuous Galerkin method, stability, error analysis, overlapping meshes.

1. Introduction

In this paper, we apply the local discontinuous Galerkin (LDG) method on overlapping meshes provided in [14] for the convection-diffusion equations

$$(1) \quad u_t + f(u)_x = (a^2(u)u_x)_x, \quad x \in [0, 1], \quad t > 0,$$

as well as its two dimensional version. We assume that $a(u) \geq 0$.

In 1973, Reed and Hill first introduced the discontinuous Galerkin (DG) method in the framework of neutron linear transportation [23]. This method gained even greater popularity for good stability, high-order accuracy, and flexibility on h-p adaptivity and complex geometry. Subsequently, Cockburn et. al. proposed in a series of papers [6, 7, 8, 9] the Runge-Kutta discontinuous Galerkin (RKDG) methods for hyperbolic conservation laws. Later, in [10], Cockburn and Shu introduced the LDG method for convection-diffusion equations motivated by successfully solving compressible Navier-Stokes equations in [1].

As in traditional LDG method, we introduce an auxiliary variable $p = A(u)_x$ with $A(u) = \int^u a(s) ds$ to represent the derivative of the primary variable u , and rewrite (1) into the following system of first order equations

$$(2) \quad \begin{cases} u_t + f(u)_x = (a(u)p)_x, \\ p = A(u)_x. \end{cases}$$

Then we can solve u and p on the same mesh by using the DG method. The LDG method shares all the nice features of the DG methods for hyperbolic equations, and it becomes one of the most popular numerical methods for solving convection-diffusion equations. However, due to the discontinuity nature of the numerical approximations, it may not be easy to construct and analyze the scheme for some special convection-diffusion equations. For example, the convection terms of chemotaxis model [19, 22] and miscible displacements in porous media [11, 12] are

products of one of the primary variable and the derivative of another one. Therefore, the upwind flux for the convection term may not be easy to obtain. One of the alternatives is to use other methods, such as mixed finite element method, to obtain continuous approximations of the derivatives, see e.g. [18]. A more general idea is to use the Lax-Friedrichs flux, see e.g. [16, 20, 28] for the error estimates for miscible displacements and chemotaxis models. The main technique is to use the diffusion term to control the convection term [24, 25, 26]. Moreover, to make the numerical solutions to be physically relevant, we have to add a sufficiently large penalty which depends on the numerical approximations of the derivatives of the primary variables [17, 20, 2]. Another possible way is to construct flux-free schemes, such as the central discontinuous Galerkin (CDG) method [21] and the staggered discontinuous Galerkin (SDG) method [4]. However, the CDG scheme doubles the computational cost as we have to solve each equation in (2) on both the primary and dual meshes twice and it is not easy to apply limiters in SDG method because it requires partial continuity of the numerical approximations.

Recently, one of the authors in this paper introduced a new LDG method on overlapping meshes [14] by solving u and p on primitive and dual meshes, respectively, hence p is continuous across the interfaces on the primitive mesh. The scheme is proved to be stable under the L^2 -norm and can be used to construct third-order maximum-principle-preserving schemes [13]. However, in some special cases, it may not enjoy the optimal convergence rates. The suboptimal convergence rate can be observed numerically if all the following three conditions are satisfied: (1) Odd order polynomials are used in the finite element space, (2) The dual mesh generated by connecting the midpoints of the primitive mesh, (3) No penalty is added to the numerical scheme. If one of the conditions is violated, the convergence rate will turn out to be optimal. Later, in [3], we used Fourier analysis to explicitly write out the error between the numerical and exact solutions and verify the optimal convergence rate for linear parabolic equations with periodic boundary conditions in one space dimension. Moreover, we also found out some superconvergence points that may depend on the perturbation constant in the construction of the dual mesh.

Both works given above are for problems with periodic boundary conditions. To implement the scheme, we need to combine the two boundary cells at the boundaries into one and find a polynomial approximation on the new cell. It is impossible to do that for general Dirichlet and Neumann boundary conditions, which are more realistic in practice, see e.g. [11, 12, 20]. In this paper, we will discuss the stability and error estimates of the new LDG methods for problems with Neumann and Dirichlet boundary conditions. The difficulty for the Neumann and Dirichlet boundary conditions is how to deal with the boundary cells of the dual mesh since two boundary cells cannot be combined. One possible way is to leave two boundary cells after generating the dual mesh, and introduce suitable numerical fluxes at the boundaries. For simplicity of presentation, we only demonstrate the proof for nonlinear parabolic equations

$$(3) \quad \begin{cases} u_t = (a(u)p)_x, \\ p = A(u)_x, \end{cases}$$

where $A(u) = \int^u a(t) dt$. The extension to general nonlinear convection-diffusion equations can be obtained following [14], hence we only demonstrate the results without proof.