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LOW REGULARITY PRIMAL-DUAL WEAK GALERKIN FINITE ELEMENT METHODS FOR ILL-POSED ELLIPTIC CAUCHY PROBLEMS

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Abstract. A new primal-dual weak Galerkin (PDWG) finite element method is introduced and analyzed for the ill-posed elliptic Cauchy problems with ultra-low regularity assumptions on the exact solution. The Euler-Lagrange formulation resulting from the PDWG scheme yields a system of equations involving both the primal equation and the adjoint (dual) equation. The optimal order error estimate for the primal variable in a low regularity assumption is established. A series of numerical experiments are illustrated to validate effectiveness of the developed theory.

Key words. Primal-dual, finite element method, weak Galerkin, low regularity, elliptic Cauchy equations, ill-posed.

1. Introduction

In this paper we consider the ill-posed elliptic Cauchy model problem: Find an unknown function u satisfying

$$
\begin{aligned}
-\nabla\cdot(a\nabla u) =& f, && \text{in} \quad \Omega,\\
u =& g_1, && \text{on} \quad \Gamma_D,\\
a\nabla u\cdot\mathbf{n} =& g_2, && \text{on} \quad \Gamma_N,
\end{aligned}
\tag{1}
$$

where $\Omega\subset\mathbb{R}^d (d=2,3)$ is an open bounded and connected domain with Lipschitz continuous boundary $\partial\Omega$, Γ_D and Γ_N are two segments of the domain boundary, $f\in L^2(\Omega)$, the Cauchy data $g_1\in H^{\frac{1}{2}}(\Gamma_D)$ and $g_2\in (H_{00}^{\frac{1}{2}}(\Gamma_N))'$, the coefficient tensor $a(x)$ is symmetric, bounded, and uniformly positive definite in the domain Ω, and $\mathbf{n}$ is the unit outward normal vector to Γ_N. The elliptic Cauchy problem is to solve partial differential equations (PDEs) in a domain where over-specified boundary conditions are given on parts of the domain boundary. The elliptic Cauchy problem is also to solve a data completion problem with missing boundary data on the remaining parts of the domain boundary.

The elliptic Cauchy problem arises in science and engineering, e.g., vibration, wave propagation, cardiology, electromagnetic scattering, geophysics, nondestructive testing and steady-state inverse heat conduction. In particular, the Cauchy problem for second order elliptic equations plays an important role in the inverse boundary value problems modeled by elliptic PDEs. Readers are referred to [4, 28, 33, 40, 1, 9, 10, 16, 17, 46, 19, 53, 7, 20, 21] and the references cited therein for details of the elliptic Cauchy problems.

There has been a long history tracing back to Hadamard [25, 43, 27, 26, 24] for the study of the elliptic Cauchy problem (1). When it comes to the case of $\Gamma_D=\Gamma_N$, Hadamard demonstrated the ill-posedness of the problem (1) by constructing an example where the solution does not depend continuously on the Cauchy data. Hadamard and others [2, 29, 30] found that a small perturbation in the data might result in an enormous error in the numerical solution for elliptic Cauchy problem.

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The Schwartz reflection principle [23] indicates that in most cases the existence of solutions for the model problem (1) can not be guaranteed for any given Cauchy data g_1 and g_2. However, [3] showed that the elliptic Cauchy problem (1) has a solution for any given Cauchy data $g_1 \times g_2 \in M$ where M is a dense subset of $H^{\frac{1}{2}}(\Gamma_D) \times [H_{00}^{\frac{1}{2}}(\Gamma_N)]'$. It is well-known that the solution of the elliptic Cauchy problem (1) (if it exists) must be unique, provided that $\Gamma_D \cap \Gamma_N$ is a nontrivial portion of the domain boundary. The Cauchy data is thus assumed to be compatible such that the solution exists. Throughout this paper, we assume that $\Gamma_D \cap \Gamma_N$ is a nontrivial portion of the domain boundary so that the solution (if it exists) of the elliptic Cauchy problem (1) is unique.

In the literature, there are two main numerical strategies developed for the elliptic Cauchy problem: (1) Tikhonov regularization is applied to the problem with missing boundary data to determine the solution; (2) A sequence of well-posed problems in the same equation is iteratively employed to approximate the ill-posed problem. [34] developed the numerical method for the elliptic Cauchy problem based on the tools of boundary integral equations, single-layer potential function and jump relations. [22] introduced an optimization approach based on least squares and Tikhonov regularization techniques. The finite element method based on an optimal control characterization of the Cauchy problem was analyzed in [14]. The stabilized finite element method [12, 11] based on a general framework involving both the original equation and its adjoint equation is applicable to a wide class of ill-posed problems where only weak continuity is necessary. More numerical methods were proposed and analyzed for the elliptic Cauchy problem including the conjugate gradient boundary element method, the boundary knot method, the alternating iterative boundary element method, the moment method, the boundary particle method, the method of level set type, and the method of fundamental solutions [35, 40, 54, 15, 41, 31, 32, 42, 53]. Other theoretical and applied work have also been developed such as regularization methods [45, 18], Steklov-Poincaré theory [6, 44, 5], minimal error methods [38, 39] and quasi-reversibility methods [8].

This paper is devoted to the development of a new primal-dual weak Galerkin finite element method for the elliptic Cauchy model problem (1). The PDWG framework provides mechanisms to enhance the stability of a numerical scheme by combining solutions of the primal and the dual (adjoint) equation. PDWG methods have been successfully applied to solve the second order elliptic equation in non-divergence form [48], the elliptic Cauchy problem [47, 49], the Fokker-Planck type equation [50], the convection diffusion equation [13, 55], and the transport equation [51, 36]. The PDWG method has the following advantages over other existing schemes: (1) it offers a symmetric and well-posed problem for the ill-posed elliptic Cauchy problem; (2) it is consistent in the sense that the system is satisfied by the exact solution (if it exists); (3) it is applicable to a wide class of PDE problems for which no traditional variational formulation is available; and (4) it admits general finite element partitions consisting of arbitrary polygons or polyhedra. The main contribution of this paper lies in two aspects: (1) the development of a new PDWG scheme that admits boundary data with low regularity due to noise or uncertainties; and (2) the establishment of a mathematical convergence theory with optimal order error estimates under low regularity assumptions for the exact solution.

Throughout the paper, we use the standard notations for Sobolev spaces and norms. For any open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary, denote by $\|\cdot\|_{s,D}$, $|\cdot|_{s,D}$ and $(\cdot,\cdot)_{s,D}$ the norm, seminorm and the inner product