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# A COMPARISON OF REGULARIZATION METHODS FOR BOUNDARY OPTIMAL CONTROL PROBLEMS

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**Abstract.** In this work we propose and compare multiple approaches for the formulation of boundary optimal control problems constrained by PDEs. In particular, we define a property of balanced regularity that is not satisfied by traditional approaches. In order to instead guarantee this property, we consider the use of other regularization terms, one involving fractional Sobolev norms and the other one based on the introduction of lifting functions. As required by the fractional norm approach, we present a semi-analytical numerical implementation of the fractional Laplacian operator. All the proposed formulations are also considered in conjunction with constraints of inequality type on the control variable. Numerical results are reported to compare all the presented regularization techniques.

**Key words.** Boundary optimal control, regularization methods, inequality constraints.

## 1. Introduction

Boundary optimal control problems are one of the most interesting classes of optimal control problems constrained by partial differential equations. In fact, the possibility of controlling the behavior of a physical system often takes place only by changing the values of certain quantities on the boundary of its domain, especially when the interior of the physical system is not accessible or no physical mechanism can be triggered inside the domain from the outside. Many works have been published both on the mathematical analysis (see [1, 2]) and the numerical approximation (see [3] and references therein) of this class of problems.

In this work we turn our attention to the mathematical formulation of boundary optimal control problems and how it affects the function space setting of optimal states and controls. Fundamental results about Sobolev spaces imply that the connection between functions defined on the domain of a PDE and their restriction to the boundary gives rise to the occurrence of *fractional-indexed* Sobolev spaces. In the context of boundary optimal control, this may lead to the presence of fractional norms and consequently fractional derivatives in the first-order necessary conditions that characterize optimal solutions.

Due to the challenges in the numerical approximation of fractional derivatives, simple optimal control techniques have been traditionally chosen in order to circumvent their presence. In many works, the control problem is simplified by resorting to integer-indexed Sobolev spaces (e.g., $H^1$ controls instead of $H^{1/2}$ controls). This approach features the drawback of a more restrictive control space than the natural space that is dictated by the range of the trace operator. In order to overcome this restriction, still without involving fractional norms, an approach based on the concept of lifting functions has been studied in the literature [4, 5, 6, 7].

To the best of our knowledge, the present work is a first attempt at encompassing multiple boundary control approaches in a unified view. In particular, the

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traditional integer-indexed approach, which is placing unnecessary restrictions on the function spaces, is compared to two formulations that bypass such limits: on one hand, the lifting function formulation; on the other, a direct approach based on a numerical approximation of the fractional Laplacian.

Fractional operators on bounded domains (and, in particular, the fractional Laplacian) can be seen as nonlocal diffusion operators [8]. For this reason, the numerical implementation of these operators is not straightforward and represents a topic of increasing interest in the scientific community. Many works have been recently published on this subject, and the interested reader can consult [9, 10]. Moreover, several articles including comparisons of the different fractional Laplacians have appeared recently, see [11, 12, 13]. A common approach for the numerical simulation of the fractional Laplacian, called spectral fractional Laplacian, is based on the Dunford-Taylor method. In particular, a direct approximation of the inverse of the fractional operator can be obtained through the so-called Balakrishnan formula and a sinc quadrature scheme. Even though this technique does not allow a direct implementation of the operator, it is advantageous from the point of view of the computational costs and the ease of implementation. The interested reader can consult [14, 15]. A different technique based on the Dunford-Taylor method, and usually called integral fractional Laplacian, can be defined using the Fourier transform. Similarly to the spectral fractional Laplacian, it relies on a sinc quadrature scheme. However, this approach allows the numerical representation of the fractional operator, instead of its inverse. The interested reader can consult [16] and references therein. Lastly, the direct numerical implementation of the real-space formula for the fractional Laplacian can be performed. This approach is known as Riesz fractional Laplacian and can be traced back to the nonlocal numerical simulations [17].

In this paper, we apply the Riesz approach to implement and test the fractional Laplacian operator in a finite element framework. Usually, a standard technique used for nonlocal simulations, including the Riesz fractional Laplacian implementations, provides the limitation of the interactions to a ball of radius $\lambda > 0$. This approach allows to reduce the computational costs and the sparsity pattern of the interaction matrix. However, this technique poses many challenges, e.g. the prescription of nonlocal analogues of boundary conditions and the uncertainty and sparsity of model parameters and data. The interested reader can consult [18, 19, 20, 21]. In this work, we introduce a semi-analytical technique for a direct implementation of the double integral, without any limitation on the interaction domain.

**Outline of the paper.** In the next Section, we introduce a class of boundary optimal control problems with particular attention to the regularization term. We present the definition of balanced regularity and consider different regularization techniques in terms of how they behave with respect to this property. These lead to different optimality systems for the proposed techniques that are presented in Section 3. Since the fractional optimal control problem gives rise to fractional operators, in Section 4 we introduce a numerical implementation of the fractional Laplacian operator. The addition of control inequality constraints is described in Section 5. Lastly, Section 6 is devoted to presenting some numerical results to compare all the proposed optimal control problems for both two- and three-dimensional simulations.