INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 2-3, Pages 369–385

REDUCED BASIS FINITE ELEMENT METHODS FOR THE KORTEWEG-DE VRIES-BURGERS EQUATION

GUANG-RI PIAO, FUXIA YAO, AND WENJU ZHAO*

Abstract. In this paper, the B-spline Galerkin finite element method and reduced order method for the Korteweg-de Vries-Burgers equation are considered. The semi-discrete and the fully discrete schemes are both provided. The reduced order model of the Korteweg-de Vries-Burgers equation by using proper orthogonal decomposition are provided. The stability and the error estimates of the corresponding schemes are then analyzed. Finally, numerical simulations are presented to show the efficiency of our proposed methods.

Key words. Korteweg-de Vries-Burgers, proper orthogonal decomposition, reduced order modeling, error analysis.

1. Introduction

In this paper, we propose numerical methods for solving the Korteweg-de Vries-Burgers (KdVB) equation: Given $\Omega = [-L, L]$, determine u such that

$$u_t + \varepsilon u u_x - \nu u_{xx} + \mu u_{xxx} = 0, \qquad x \in \Omega, 0 \leq t \leq T, \tag{1}$$

$$u(-L, t) = u(L, t) = 0, \qquad x \in \Omega, 0 \leq t \leq T, \tag{2}$$

$$u_x(-L, t) = u_x(L, t) = 0, \qquad x \in \Omega, 0 \leq t \leq T, \tag{3}$$

$$u\left(x, 0\right) = u_0\left(x\right), \qquad x \in \Omega, \tag{4}$$

where $\varepsilon, \nu, \mu \in \mathbb{R}$ are real positive parameters with $\varepsilon\nu\mu \neq 0$.

For the past several decades, many mathematicians and physicists have paid great attention to such kind of problems. The KdVB equation is one of the most important non-linear partial differential equations, which was developed by Su and Gardner [25] to describe the weak effects of dispersion, dissipation and nonlinearity of wave propagation in a liquid-filled elastic tube. For the parameter $\nu = 0$, (1)-(4) will be reduced to the KortewegDe Vries (KdV) equation which has been used to describe the dynamical effects, i.e., ion sound, plasma shock wave [10, 23, 24, 30]. For the parameter $\mu = 0$, (1)-(4) will be simplified to the Burgers equation that has a widely physical application in many fields, i.e., shock wave propagation, turbulence flow, etc. Some theoretical regularities such as the existence, uniqueness, stability of KdV-type equations have been studied in [1, 9, 14, 22], etc. The KdVB equation incorporates the properties of the KdV equation and Burgers equation which are of great interest to be studied, and has high research value in applied mathematics.

Many numerical methods have been already studied for the KdV-type equations, i.e., the finite element method [11, 28] and finite difference discontinuous Galerkin method [12], finite difference method [16, 27], etc. In this paper, we will numerically analyze and simulate the KdVB equation. Numerical simulations of nonlinear systems are relatively expensive with respect to both the storage and the computational complexity, where the iterative methods for the nonlinear system are usually required. To efficiently solve this kind of problems, many reduced-order modeling

Received by the editors August 30, 2021 and, in revised form, March 06, 2022.
2000 *Mathematics Subject Classification.* 35G61, 65D07, 65M15, 65M60.
*Corresponding author. Email: zhaowj@sdu.edu.cn.

techniques are developed. One of the popular reduced order methods at least for the applications is the proper orthogonal decomposition (POD) analysis. The POD techniques combined with the Galerkin methods have been widely used to formulate the reduced order modelings for dynamic systems [2, 5, 6, 15, 17, 20, 21], which can provide precise approximation with reduced number of degrees of freedom. Moreover, the induced lower dimensional models alleviate the computational load and memory requirements [3]. In this paper, the approaches to efficiently handle the nonlinear terms and generate the snapshots are referred to the techniques for the reduced-order modeling for the Navier–Stokes equations [5, 6, 29]. Similarly to the fourth order equations [7], the third-order KdVB equation inherits higher regularity than that for the second order partial differential equation. In turn, the usual C^0 finite element basis with less regularity for the second order partial differential equation is usually not feasible for the KdVB equation.

In this paper, the quadratic B-spline basis with continuous first derivative is used The main contribution of paper is to perform theoretical analyses of the quadratic B-spline Galerkin finite element approximation for the KdVB equation and its related reduced order modeling based on the POD Galerkin finite element approximation.

This paper is organized as follows. In following part of Section 1, we introduce the notation and preliminaries which are used throughout the paper. In Section 2, the Galerkin finite element methods are provided. The semidiscrete and fully discrete schemes are analyzed. In Section 3, we obtain the reduced dimension surrogate model of the KdVB equation by using proper orthogonal decomposition technique. We also indicate the error between the reduced model solution and its regular solution. Numerical simulations are presented in the final section.

1.1. Notation and Preliminaries. We use standard notation for the function spaces. For any integer $k \geq 0$, $H^k(\Omega)$ denotes the Sobolev space on Ω associated with inner product $(\cdot,\cdot)_{H^k}$ and norm $\|\cdot\|_k$. On the space $L^2(\Omega) := H^0(\Omega)$, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the L^2 inner product and norm, respectively. $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The Sobolev space $H_0^k(\Omega)$ with $k = 2$ is then defined as

$$H_0^k(\Omega) = \left\{ u \in H^k(\Omega) : \partial_x^j u(x) = \partial_x^j u(x+L) = 0, j = 0, \ldots, k-1 \right\}, \tag{5}$$

where $\partial_x^j u(x)$ represents the jth derivative in the sense of distribution with respect to x of a function u. Let $C([0,T];H^k)$ be the space of all continuous functions $u : [0,T] \to H^k(\Omega)$ with $\|u\|_{C([0,T];H^k)} = \max_{0\leq t\leq T} \|u(t)\|_k < \infty$. Denote by $L^2([0,T];H^k)$ the space of square integrable functions $u : [0,T] \to H^k(\Omega)$ with $\|u\|^2_{L^2([0,T];H^k)} = \int_0^T \|u(t)\|_k^2 dt < \infty$. To be brief, we set $u_x := \partial_x u$ and $u_{xx} := \partial_x^2 u$. The variational form of (1)-(4) is derived by multiplying (1) with a function $v(x) \in H_0^2(\Omega)$ and integrating by parts on Ω. The weak formulation of (1)-(4) is then written as

$$(u_t, v) - \frac{\varepsilon}{2}\left(u^2, v_x\right) + \nu\left(u_x, v_x\right) - \mu\left(u_{xx}, v_x\right) = 0 \tag{6}$$

for almost every $t \in [0,T]$.

Let $M \in \mathbb{N}^+$ be a positive integer. Define the spatial mesh $\mathcal{T}_h$ with mesh size $h = 2L/M$. The grid points are denoted as $x_j = -L + jh, j = 0, 1, \ldots, M$ with subintervals $I_j = [x_j, x_{j+1}], j = 0, 1, \cdots, M-1$. Let $P_r(I)$ denote the space of polynomials on the interval I of degree no greater than $r \in \mathbb{N}^+$. We seek a discrete approximation u_h to the solution of (1)-(4) such that for all $t \in [0,T]$, $u_h(t)$ belongs