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A WEIGHTED LEAST-SQUARES FINITE ELEMENT METHOD FOR BIOT'S CONSOLIDATION PROBLEM

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Abstract. This paper examines a weighted least-squares method for a poroelastic structure governed by Biot's consolidation model. Quasi-static model equations are converted to a first-order system of four-field, and the least-squares functional is defined for the time discretized system. We consider two different sets of weights for the functional and show its coercivity and continuity properties, from which an a priori error estimate for the primal variables is derived. Numerical experiments are provided to illustrate the performance of the proposed method.

Key words. Weighted least-squares finite element method, Biot's consolidation model.

1. Introduction

Biots consolidation model provides a general description of the mechanical behavior of a poroelastic medium and is frequently used in a wide range of applications in geomechanics, bioengineering, environmental engineering, and various other science and engineering areas. The model is based on the equation of linear elasticity for a solid matrix and Darcys law for the fluid flow through a porous matrix [2, 3]. Generally, solutions of the model are approximated by numerical methods since the analytical solution can only be derived under the assumption of special conditions [32, 36]. Finite element methods are commonly used in simulations. There have been various finite element methods proposed for the poroelasticity, including mixed finite element methods [15, 26, 28, 31, 35], discontinuous Galerkin finite element methods [13, 19], least-square methods [20, 21, 33], and hybrid methods [22, 29, 34] and a decoupling approach [14].

Problems for which solutions are smooth can be solved by standard finite element discretization. However, a finite element solution may have non-physical oscillations, known as pressure locking, if it displays some high-pressure gradient [15, 16, 17]. For example, pressure locking can occur when finite element spaces are not compatible. Some hybrid finite element methods [29, 35] have been proposed to overcome this issue. Another locking phenomenon called elasticity locking is observed when one of the Lamé coefficients becomes large, with the Poisson ratio approaching 0.5 [30].

The difficulties caused by the incompatibility of the spaces can be avoided by least-squares finite element methods. One of the main advantages of least-squares finite element methods is that no *inf-sup* condition is required between finite element spaces. Such flexibility makes the least-squares approach appealing for the finite element approximation of differential equations with multiple variables. This work aims to study a weighted least-squares (WLS) functional defined for the time discretized quasi-static Biot model and compares numerical solutions by the WLS finite element method with various weights. The WLS functional is defined using the L^2-norm of the equation residuals multiplied by appropriately adjusted weights. Various developments have been reported for WLS finite element methods applied

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to flow problems. Bochev and Gunzburger [4] developed a mesh-dependent weight of the WLS functional for Stokes flows based on the Agmon-Douglas-Nirenberg (ADN) approach. Weighted-norm least-squares methods were considered for problems with corner or coefficient singularities in [1, 10, 23]. In addition, Lee and Chen [24] applied a nonlinear weight to least-squares functional for Stokes equations, and this approach was further developed for non-Newtonian viscoelastic fluids [12, 25].

While extensive work on finite element approximations and analysis have been devoted to the Biot model, only a few studies of least-squares finite element methods have been carried out for the model [20, 21, 33]. In [20], Korsawe and Starke developed a four-field mixed least-squares finite element method for the quasi-static model with a simplified mass equation and unified modeling parameters. They defined the least-squares functional for the stationary case that arises at each time step to solve the temporal discretized model and proved the coercivity and continuity of the functional. In [21], Korsawe et al. numerically studied the Biot model and compared least-squares results with the standard Galerkin method results. The authors discussed the accuracy of stress and flux variables approximated directly in the least-squares method, pointing out that the additional unknowns increase the degree of freedom of the discretized problem compared to the Galerkin method. Tchonkova et al. discussed the mixed least-squares method for the poroelasticity problem of four-field and approximated solutions using linear continuous polynomials for all variables on triangle elements [33]. However, in [21, 33], no weights were considered for the least-squares functionals.

This work further develops the least-squares approach and analysis presented in [20] for the full quasi-static model with all modeling parameters. We consider a WLS finite element method in a similar setting presented in [20]; the least-squares functional is defined for the four-field modeling equations discretized in time, where a weight for each term of the functional is appropriately chosen. Some of those weights need to be dependent on the time step for the analysis of the WLS functional. The choice of different sets of weights is also addressed. The WLS functional is then analyzed for the coercivity and continuity properties. It is demonstrated that the use of weights for the functional is helpful for the analysis and improves the accuracy of numerical solutions. Further, we extend the implementation to the intracranial pressure simulation [18].

The rest of this paper is organized as follows. Section 2 presents the model equations and the least-square functional. Section 3 introduces the WLS functional and the analysis for the functional. Section 4 presents finite element spaces and error estimation of finite element approximations. Section 5 provides two numerical examples, where numerical solutions by different sets of weights are compared, and finally, conclusions follow in Section 6.

2. Model equations and least-squares functional

Let Ω be a bounded, connected domain in $\mathbb{R}^d$, $d = 2, 3$ with the Lipschitz boundary $\partial\Omega$. Consider the quasi-static poroelastic system represented by the Biot model [2]:

$$\nabla \cdot \mathbf{u} + \frac{\partial}{\partial t}(c_s p + \alpha \nabla \cdot \boldsymbol{\eta}) = f_s \text{ in } \Omega, \tag{1}$$

$$\mathbf{u} + K \nabla p = \mathbf{0} \text{ in } \Omega, \tag{2}$$

$$-2\mu \nabla \cdot \varepsilon(\boldsymbol{\eta}) - \lambda \nabla(\nabla \cdot \boldsymbol{\eta}) + \alpha \nabla p = \mathbf{f}_b \text{ in } \Omega, \tag{3}$$