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A FINITE DIFFERENCE METHOD FOR ELLIPTIC PROBLEMS WITH IMPLICIT JUMP CONDITION

FUJUN CAO, DONGFANG YUAN, ZHIQIANG SHENG, GUANGWEI YUAN*, AND LIMIN HE

Abstract. In this paper linear elliptic problems with imperfect contact interface are considered, and a second order finite difference method is presented for linear problems, in which implicit jump condition are imposed on the interface. Then, the stability and convergence analysis of the FD scheme are given for the one-dimensional elliptic interface problem. Numerical examples are carried out for the elliptic problems with imperfect contact interfaces, and the results demonstrate that the scheme has second order accuracy for elliptic interface problems of implicit jump conditions with single and multiple imperfect interfaces.

Key words. Implicit jump conditions, elliptic interface problem, imperfect contact.

1. Introduction

Interface problems occur in many multi-physics and multi-phase applications in science and engineering, particularly for free boundary/moving interface problems, for examples, the modeling of the Stefan problem of solidification process and crystal growth, composite materials, multi-phase flows, cell and bubble deformation, and many others. To be simple to expression, we consider the interface problems in multi-material heat transfer process. According to the different jump conditions, the interface problems can be divided into two main categories: (1) Perfect contact, that is, the contact between the two objects is perfect, which means that the temperature and normal heat flux are continuous on the interface. (2) Imperfect contact, for example, there are weakly conductive thin films or interlayers between the two objects, so that temperature or normal heat flux is discontinuous across the interface. In practice, an equivalent boundary condition is often presented on the thin layer, namely the interface jump (or connection) condition. When the contact interface is not perfect, the jump condition on the interface can be roughly divided into the following classes.

(1) The first class of imperfect interface condition is that jump sizes are given [15, 16, 17, 21, 22], which can be named as explicit jump condition for the sake of convenience and are shown as follows:

$$(1)\qquad \begin{cases} [u] = u^+ - u^- = h_1(x), \quad \text{on} \quad \Gamma, \\ [\kappa \frac{\partial u}{\partial \vec{n}}] = \kappa^+ \frac{\partial u^+}{\partial \vec{n}} - \kappa^- \frac{\partial u^-}{\partial \vec{n}} = h_2(x), \quad \text{on} \quad \Gamma, \end{cases}$$

where $h_1(x)$ and $h_2(x)$ are given functions.

(2) The second class of imperfect interface condition is that the jump size of temperature is proportioned to flux, which can be named as implicit jump condition

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*Corresponding author.

and be written in the following form [1, 2, 3, 9, 11, 12, 36, 58]

$$(2) \qquad \begin{cases} [u] = u^+ - u^- = \lambda\kappa^- \frac{\partial u^-}{\partial \vec{n}}, & \text{on} \quad \Gamma, \\ [\kappa \frac{\partial u}{\partial \vec{n}}] = \kappa^+ \frac{\partial u^+}{\partial \vec{n}} - \kappa^- \frac{\partial u^-}{\partial \vec{n}} = 0, & \text{on} \quad \Gamma, \end{cases}$$

where Γ is a curve which divides the region Ω into two non-intersected subregions Ω^+ and Ω^-, $\Omega = \Omega^- \cup \Omega^+ \cup \Gamma$. $\vec{n}$ is the outer unit normal vector of the interface Γ in Ω^-. κ^- and κ^+ represent the material conduction coefficients on Ω^- and Ω^+, respectively.

In this paper we consider the problems with the interface conditions, where the jumps of temperature are related to the normal heat fluxes. The second class implicit connection condition of the imperfect interface can be used to describe the heat conduction problem of two objects with imperfect contact [1]. If there is a interlayer with thickness δ and the thermal conductivity is ϵ between two objects, and when $\delta \to 0$, $\epsilon/\delta \to const = \lambda$, then the interlayer is degenerated into a sharp interface. In the implicit jump conditions, the jumps of physical quantities are unknown and proportional to the flux across the interface. The implicit connection condition has a clear physical meaning. Moreover, it can be used to describe the problem of temperature discontinuity between gas and cooling solid surface [4]. In addition, it is appeared in some other applications, such as the effective thermal conductivity of composite materials [5], the dielectric heat conduction problem of solid spherical particles dispersed in the continuous phase [6], the interface problem with thermal resistance between the composite and the discrete components [7]. In the problem of steady thermal diffusion in a two component nonhomogeneous conductor with contact resistance, the flow of heat through the material interface is also considered to be proportional to the jump of the temperature field [9, 10, 13]. The solution to an imperfect interface problem, therefore, typically is non-smooth or even discontinuous across the interfaces. It is necessary to study accurate and robust numerical methods for these elliptic interface problems.

When the jump sizes along the interface are known explicitly, (say $[u] = h_1$, $[\kappa\partial u/\partial\vec{n}] = h_2$, with given h_1 and h_2), there are various numerical approaches, such as immersed interface method (IIM) [15, 16, 17, 19, 20, 23], immersed finite volume method (IFVM) [24, 25, 26] and immersed finite element (IFE) methods [27, 28, 30, 31, 34, 37, 38, 45, 56], and they are presented to effectively handle the explicit jump conditions. Since the pioneer work by [15], the immersed interface method (IIM) has becoming increasingly popular for elliptic interface problems. The original IIM achieve uniformly second order accuracy, and a key feature is that computational stencils for irregular points are modified such that the information on the boundary is used exactly where grid lines intersect the immersed boundary. Recently, there have been many further developments and analysis in various aspects of the immersed interface methods [16, 17, 19, 21, 22]. Among these developments, Li and Ito [17] constructed a fourth-order accurate finite difference method for interface problems which produces a large sparse linear system with M-matrix in several-dimensions and is coupled with a multigrid solver achieving fast convergence of the linear solver. Wiegmann and Bube [21] developed an explicit-jump immersed interface method for some special cases, where the explicit jump conditions of physical quantity and its derivatives ($[u]$, $[u_x]$, $[u_{xx}]$, etc.) are known. Mittal et al. [23] use standard finite difference formulas at grid points near the interface using interfacial points also as one of the nodes and the Lagrange polynomial interpolation is then used to find the unknown values at interfacial points. The proposed scheme is derived for general elliptic interface problems with explicitly known functions