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A MULTISCALE PARALLEL ALGORITHM FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATION IN COMPOSITE MEDIA

FANGMAN ZHAI AND LIQUN CAO

Abstract. This paper studies the multiscale algorithm for parabolic integro-differential equations in composite media combining with Laplace transformation. The new contributions reported in this study are threefold: the convergence estimates with an explicit rate for the multiscale solutions of the equations in general domains are proved, the boundary layer solution is defined and the multiscale finite element algorithm which is suitable for parallel computation is presented. Numerical simulations are then carried out to validate the theoretical results.

Key words. Parabolic integro-differential equation, the multiscale asymptotic method, Laplace transformation, composite media.

1. Introduction

In this paper, we consider the parabolic integro-differential equations with rapidly oscillating coefficients as follows:

$$(1)\quad \begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big) - \dfrac{\partial}{\partial x_i}\displaystyle\int_0^t \beta(t-s)a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,s)}{\partial x_j}\,\mathrm{d}s \\ \qquad = f(x,t), \quad (x,t)\in\Omega\times(0,T), \\ u^\varepsilon(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = \bar{u}_0(x), \quad x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 1$ is a bounded convex polygonal domain or a bounded smooth domain with a periodic microstructure. Here ε is a small periodic parameter. $a_{ij}(\frac{x}{\varepsilon})$, $\beta(t)$, $f(x,t)$, $g(x,t)$ and $\bar{u}_0(x)$ are given functions. We note that here and in the sequel the Einstein summation convention is adopted on repeated indices.

Let $\xi=\varepsilon^{-1}x$ and we make the following assumptions:

(A_1) $a_{ij}(\xi)$, $i,j=1,2,\cdots,n$ are 1-periodic in ξ.

(A_2) $a_{ij}=a_{ji}$, $\gamma_0|\eta|^2\leq a_{ij}(\xi)\eta_i\eta_j\leq\gamma_1|\eta|^2$, $\gamma_0,\gamma_1>0$, $\forall(\eta_1,\eta_2,\cdots,\eta_n)\in\mathbb{R}^n$, where γ_0,γ_1 are constants independent of ε.

(A_3) $a_{ij}\in L^\infty(\Omega), \beta\in L^1(0,T), f\in L^2(0,T;L^2(\Omega)), g\in L^\infty(0,T;H^{\frac{1}{2}}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)$.

(A_4) Let $Q=(0,1)^n$ be the reference cell and let Q' be a bounded domain in $\mathbb{R}^n$ with a $C^{1,\mu}$ boundary, $0<\mu<1$. Let $Q\subset\subset Q'$, $\bar{Q}'=\bigcup_{m=1}^L\big(\overline{D}_m\big)$ be a union of some subdomains, where for each D_m, $\partial D_m\in C^{1,\mu}$ and $D_m\cap D_k=\emptyset$ for $m\neq k$. Assume $a_{ij}\in C^\gamma(\overline{D}_m), i,j=1,2,\cdots,n,$ for some $0<\gamma<1, m=1,2,\cdots,L$, where L, μ, γ are constants independent of ε (cf. [16]).

Remark 1.1. *Under assumptions (A_2)-(A_3), the well-posedness for the problem* (1) *can be established (see, e.g.,* [10, 12, 27]*).*

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The problem (1) has wide applications in heat conduction with memory effects, nuclear reactor dynamics, blow-up problems in composite material or in porous media(see,e.g.,[20, 22, 33, 35] and the references therein). For a special choice of the kernel $\beta(t) = t^{\alpha-1}/\Gamma(\alpha)$ $(0 < \alpha < 1)$, the integral term of (1) is actually the Riemann-Liouville fractional integral of the function $\dfrac{\partial}{\partial x_i}\Big(a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big)$ (see, e.g., [25, 28]). In this case, (1) is a linear integro-differential equation of fractional order. These kinds of equations describe anomalous diffusion processes and the wave propagation in viscoelastic materials (cf. [13, 17, 19, 26]), which have attracted considerable attention of researchers in recent years (see, e.g.,[7, 8]). As a parameter $\varepsilon > 0$ is small enough, the direct numerical simulation for the problem (1) is a hard work because it would require a very fine mesh, a very small time step and the massive storage of the numerical solutions at all time steps.

All kinds of homogenization methods are utilized to solve the partial differential equations and the integro-differential equations with rapidly oscillating coefficients (e.g., periodic, almost periodic, quasi-periodic and non-periodic). For instance, about the homogenization methods concerning linear parabolic equations, we refer to Bensoussan et al. [2] and Sanchez-Palencia [29] for periodic cases and to Colombini and Spagnolo [6] for general non-periodic cases. Zhikov et al. [37] studied parabolic operators with almost periodic coefficients and derived convergence results for the asymptotic homogenization. The homogenization method for the nonlinear parabolic equations can be found in [24, 30].

Numerous simulation results have shown that the numerical accuracy of the homogenization methods may not be satisfactory when ε is not small enough (see, e.g. [3, 32]). So we need to seek the multiscale methods to improve the numerical accuracy. Bensoussan et al. [2] investigated the first-order multiscale asymptotic method for the linear parabolic equations with oscillating periodic coefficients. For the higher-order multiscale method for the linear parabolic equations, we refer to Allegretto et al. [1]. Huang et al. [15] studied the multiscale method and obtained the strong convergence results with an explicit rate for a kind of nonlinear parabolic equations. On the other hand, various multiscale numerical approaches are also available. Hou [14] and his collaborators first presented the multiscale finite element method (MsFEM) for elliptic equations in composite materials or in porous media. Efendiev et al. [4, 5, 9] developed the multiscale finite element methods, for instance, GMsFEM, CEM-GMsFEM, NLMC and so forth. Ming and Zhang [21] proposed the heterogeneous multiscale method(HMM) for parabolic problems.

To the best of our knowledge, few results of the multiscale methods for the problem (1) have been reported. In this paper, we will study the multiscale analysis and computation for the problem (1). We notice that the classical multiscale asymptotic method fails in the study of the problem (1), due to the integro-differential term in the equation. Bensoussan et al. [2] first employed the Laplace transformation to investigate the homogenized method for an integro-differential equation of hyperbolic type. Wang et al. [32] combined the Laplace transform method with the multiscale method to study the coupled thermoelastic system in composite materials and derived the first strong convergence results with an explicit rate of the second-order multiscale solutions for the coupled thermoelastic system. In our recent work [36], we used the Laplace transformation to discuss the multiscale analysis and computation for the dual-phase-lagging equation in composite materials. Inspired by the above ideas, in this paper we use the Laplace transform method to discuss the multiscale analysis and algorithm for the problem (1). The procedure of our method is briefly described. First, we employ the Laplace transform to transfer the original