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AN IMMERSED CROUZEIX-RAVIART FINITE ELEMENT METHOD FOR NAVIER-STOKES EQUATIONS WITH MOVING INTERFACES

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Abstract. In this article, we develop a Cartesian-mesh finite element method for solving Navier-Stokes interface problems with moving interfaces. The spatial discretization uses the immersed Crouzeix-Raviart nonconforming finite element introduced in [29]. A backward Euler full-discrete scheme is developed which embeds Newton's iteration to treat the nonlinear convective term. The proposed IFE method does not require any stabilization terms while maintaining its convergence in optimal order. Numerical experiments with various interface shapes and jump coefficients are provided to demonstrate the accuracy of the proposed method. The numerical results are compared to the analytical solution as well as the standard finite element method with body-fitting meshes. Numerical results indicate the optimal order of convergence of the IFE method.

Key words. Navier-Stokes, interface problems, nonconforming immersed finite element methods, moving interface.

1. Introduction

Multi-phase immiscible incompressible flows with embedded interfaces are widely present in many physical phenomena. The related simulations appear in many branches of science and engineering, such as fluid dynamics, biology, medical sciences, and geology [9, 12, 13, 41, 49], to name just a few. The dynamics of the two-phase (or multi-phase) flows are governed by the well-known Navier-Stokes (NS) equations, or Stokes equations for creeping flows, along with the enforcement of jump conditions at interfaces. Physical parameters of the flows, such as density and viscosity coefficients, are usually discontinuous across the fluid interface [1, 9, 17, 29, 49].

In this article, we consider a two-dimensional interface problem that arises in a two-phase flow governed by the NS equation. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain separated by an interface $\Gamma(t)$ into two disjoint subdomains $\Omega^-(t)$ and $\Omega^+(t)$. Consider the following unsteady NS equation (NSE) in the velocity-stress-pressure form:

$$\boldsymbol{u}_t - \nabla \cdot \sigma(\boldsymbol{u}, p) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega^-(t) \cup \Omega^+(t) \times [0,T], \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \times [0,T], \tag{1b}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega \times [0,T], \tag{1c}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0,\ p(\boldsymbol{x},0) = p_0 \quad \text{in } \Omega, \tag{1d}$$

where $\boldsymbol{u}$ represents the velocity field and p represents the pressure. The stress tensor $\sigma(\boldsymbol{u}, p)$ is defined by

$$\sigma(\boldsymbol{u}, p) = 2\mu\boldsymbol{\epsilon}(\mathbf{u}) - p\boldsymbol{I},$$

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where $\boldsymbol{\epsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t)/2$ is the strain tensor and $\boldsymbol{I}$ is the identity tensor. The viscosity coefficient $\mu(\boldsymbol{x})$ is discontinuous across the interface $\Gamma(t)$, which is a positive piecewise-constant function defined by

$$\mu(\boldsymbol{x}) = \begin{cases} \mu^- & \text{if } \boldsymbol{x} \in \Omega^-(t), \\ \mu^+ & \text{if } \boldsymbol{x} \in \Omega^+(t). \end{cases} \tag{2}$$

Across the interface, the following homogeneous velocity and stress jump conditions are enforced

$$[\boldsymbol{u}]_\Gamma = \mathbf{0} \text{ on } \Gamma(t), \tag{3a}$$

$$[\sigma(\boldsymbol{u}, p)\mathbf{n}]_\Gamma = \mathbf{0} \text{ on } \Gamma(t), \tag{3b}$$

where the jump $[\cdot]_\Gamma$ is defined by $[\mathbf{v}]_{\boldsymbol{\Gamma}} := \mathbf{v}^+|_{\boldsymbol{\Gamma}} - \mathbf{v}^-|_{\boldsymbol{\Gamma}}$, and $\mathbf{n}$ is the unit normal vector to the interface Γ pointing from Ω^- to Ω^+. We also note that when $\mu(\boldsymbol{x})$ is a piecewise constant, due to the divergence condition (1b), the momentum equation (1a) can be written as

$$\boldsymbol{u}_t - \mu\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } (\Omega^-(t) \cup \Omega^+(t)) \times [0, T]. \tag{4}$$

Under this framework, the stress jump condition (3b) can be modified as follows

$$[(\mu\nabla \boldsymbol{u} - p\boldsymbol{I})\,\boldsymbol{n}]_\Gamma = \mathbf{0}. \tag{5}$$

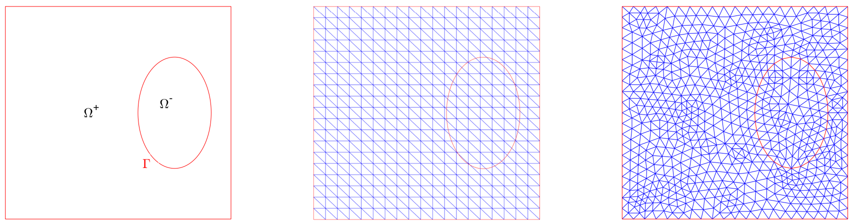

FIGURE 1. a domain with an interface (left), a non-body-fitting mesh (middle) and a body-fitting mesh (right).

The Navier-Stokes equations sans an interface are widely studied in the context of finite element methods, including classical finite element methods [16, 18], discontinuous Galerkin (DG) finite element methods [10, 15, 45, 47], and weak Galerkin finite element methods [27, 46, 51]. Those finite element methods can be extended to solve pertinent interface problems provided that body-fitting meshes are employed. However, such body-fitting restriction may hinder the efficiency in solving interface problems with evolving interfacial geometries and locations because the mesh has to be generated repeatedly according to each interface configuration. Many numerical methods based on interface-independent meshes have been developed, such as cut finite element method (CutFEM) [5, 6, 7, 42], immersed interface method (IIM) [33], extended finite element method (XFEM)[14], partition of unity finite element method (PUFEM) [44] and matched interface and boundary (MIB) [53] method. These unfitted-mesh numerical methods employ modified weak formulations or revised finite element functions around the interface to capture the interfacial jump behaviors. We refer the readers to [11, 17, 26, 43] for CutFEM, [34] for IIM, [48] for XFEM, [4] for PUFEM, and [52] for MIB applied to NS moving interface problems. An illustration of a body-fitting mesh and a non-body-fitting mesh is given in Figure 1.