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A LEAST-SQUARES STABILIZATION VIRTUAL ELEMENT METHOD FOR THE STOKES PROBLEM ON POLYGONAL MESHES

YANG LI, CHAOLANG HU AND MINFU FENG*

Abstract. This paper studies the virtual element method for Stokes problem with a least-squares type stabilization. The method cannot only circumvent the Babuška-Brezzi condition, but also make use of general polygonal meshes, as opposed to more standard triangular grids. Moreover, it is suitable for arbitrary combinations of the velocity and pressure, including equal-order virtual element. We obtain the corresponding energy norm error estimates and L^2 norm error estimates for velocity. Finally, a series of numerical experiments are performed to verify the method has good behaviors.

Key words. Virtual element method, Stokes problem, Least-squares stabilization.

1. Introduction

Recently, there has been increasing interest in developing numerical methods that can make use of non-traditional meshes (e.g., polygons and polyhedra). Indeed, the use of polygonal meshes brings a number of advantages, including more efficient approximation of geometric data features, better domain meshing capabilities, more robustness to mesh deformation, and others. Several numerical methods for polygon meshes have been proposed, such as Hybrid High-Order method [1], Mimetic Finite Difference(MFD) method [2], Weak Galerkin method [3–5], Virtual Element method(VEM) [6].

Among them, the VEM follows the main ideas of MFD method [7–11]. By avoiding the explicit expression of basis function and the use of quadrature formula, it can handle the problem on general polygonal meshes. Indeed, virtual element space contains space of polynomials, with addition of suitable non-polynomial functions. The polynomial part is used to insure the consistency of the scheme, while the non-polynomial part is used to insure the stability, see [6, 12] for details. So far, the VEM has been used to solve various problems, such as elliptic [13] and parabolic equations [14], linear elastodynamics [15, 16], Cahn-Hilliard equation [17], Stokes [18–28] and Navier-Stokes(N-S) equation [29]. In addition, several different numerical methods have been analysed, such as H(div) and H(curl)-conforming VEM [30], discontinuous Galerkin VEM [31], nonconforming VEM [32, 33], mixed VEM [34].

All these mixed virtual element methods are constructed in such a way that only the velocity is approximated by a virtual element space while the pressure discretization is based on a traditional finite element space. The basis of their analysis is that the Babuška-Brezzi(B-B) condition must be satisfied, which is not easy in some cases. In order to deal with this problem, how to combine VEM

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*Corresponding author: fmf@scu.edu.cn.

with stabilization method [35,36] is an interesting problem. Recently, a stabilized VEM for N-S problem has been proposed in [37], although numerical tests show this method is stable, there is no theoretical analysis. In [38,39], the authors proposed a SUPG-stabilized conforming/nonconforming VEM for convection-diffusion-reaction problem. Moreover, a projection-based stabilized VEM for the Stokes problem has been considered in [40], which constitutes a low-cost solver. Nevertheless, it's only weakly consistent.

As we know least-squares technique has been frequently used to simulate solutions of partial differential equations. Its numerical stability is not sensitive to the choice of spaces or meshes. In addition, it's not subject to the B-B condition and leads to a symmetric and positive definite algebraic system. We believe the attractive characteristics of the VEM and least-squares method should remain if both of them are combined. This motivates us to consider the least-squares stabilization VEM for solving the Stokes problem. The approximation of mixed formulations is suitable for arbitrary finite element combinations of the two variables. That is, the method is applicative when velocity is approximated by virtual element, regardless of pressure is approximated by virtual element or traditional finite element. In particular, when both velocity and pressure are approximated by classical finite element pairs, the method becomes that proposed in [36] by Franca et al. We establish the stability and corresponding error estimates, including the energy norm and the L^2 norm for velocity. And several numerical tests confirm the theoretical convergence results.

The structure of the paper is as follows. Section 2 states the Stokes problem and its weak formulation. In Section 3, we outline the virtual element discretization of the problem. Section 4 considers the stability of the proposed method. The error estimates in the energy norm and the L^2 norm for velocity are investigated in Section 5. Section 6 gives some numerical tests which confirm the theoretical results. Finally, we present short conclusions.

Notation Throughout the paper C will denote a common constant, that is independent of the mesh size h. We use the standard notations for Sobolev space, norm, and semi-norm. More specifically, given bounded Lipschitz domain $D \subset \mathbb{R}^d$, we define by $W^{k,p}(D)$ the space of all L^p integrable functions in D whose weak derivatives less than or equal to order k are also L^p integrable. For $p = 2$, we denote $H^k(D) := W^{k,2}(D)$, and we utilize $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ to represent the corresponding norm and semi-norm, respectively. In the Sobolev space $H^k(D)$, while $(\cdot,\cdot)_{k,D}$ denote the corresponding inner product, and the standard $L^2(D)$ inner product is denoted by $(\cdot,\cdot)_D$ with corresponding norm $\|\cdot\|_{0,D}$.

2. The continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a convex, bounded polygonal domain with homogeneous Dirichlet boundary condition. We consider the Stokes problem:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{u}$ and p denotes the velocity and pressure fields, respectively. And $\mathbf{f}$ denotes the external force, ν is a constant that represents the viscosity.