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A DECOUPLED, PARALLEL, ITERATIVE FINITE ELEMENT METHOD FOR SOLVING THE STEADY BOUSSINESQ EQUATIONS

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Abstract. In this work, a decoupled, parallel, iterative finite element method for solving the steady Boussinesq equations is proposed and analyzed. Starting from an initial guess, an iterative algorithm is designed to decouple the Naiver-Stokes equations and the heat equation based on certain explicit treatment with the solution from the previous iteration step. At each step of the iteration, the two equations can be solved in parallel by using finite element discretization. The existence and uniqueness of the solution to each step of the algorithm is proved. The stability analysis and error estimation are also carried out. Numerical tests are presented to verify the analysis results and illustrate the applicability of the proposed method.

Key words. Steady Boussinesq equations, decoupled parallel iterative algorithm, finite element method, error analysis.

1. Introduction

The system of Boussinesq equations is an important model in fluid dynamics, describing incompressible flow driven by heat difference, namely the natural convection phenomenon. The typical examples of the convection can be found in nature, such as the ocean flow driven by temperature difference, the ventilation in a room, and the ground water system (see [20, 48, 50, 63, 75, 80]). In engineering, free convection is exploited in numerous applications, such as double-glazed windows, cooling in small electronic devices, building insulation, and environmental transport problems (see [4, 21, 37, 47, 49, 78]).

In the Boussinesq model, the density of the fluid is kept constant and the gravitational force depends on the temperature. In this approximation, the fluid and the temperature are coupled by two terms. The first one is a buoyancy term, which linearly depends on the temperature and acts in the direction opposite to the gravity, in the stationary incompressible Navier-Stokes equations of the fluid variables. The second one is a convective term, which is based on the velocity of the fluid, in the convection-diffusion equation of the temperature variable.

In this work, the stationary Boussinesq equations are considered:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} - Pr\Delta\mathbf{u} + \nabla p = PrRa\hat{\mathbf{g}}\theta + \boldsymbol{\gamma}_1, \text{ in } \Omega, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0}, \text{ on } \partial\Omega, \tag{3}$$

$$\mathbf{u}\cdot\nabla\theta - k_0\Delta\theta = \gamma_2, \text{ in } \Omega, \tag{4}$$

$$\theta = 0 \text{ on } \Gamma_0, \ \nabla\theta\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\setminus\Gamma_0, \ |\Gamma_0| \neq 0. \tag{5}$$

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Here Ω is a bounded domain in R^d with Lipschitz continuous boundary $\partial\Omega$, where $d = 2, 3$ is the space dimension. Γ_0 is part of $\partial\Omega$ with its measure $|\Gamma_0| \neq 0$. $\mathbf{u}$ is the fluid velocity, p the pressure, and θ the temperature. Furthermore, $\boldsymbol{\gamma}_1$ and γ_2 are the given force functions in $[H^{-1}(\Omega)]^d$ and $H^{-1}(\Omega)$, respectively. Pr and Ra are Prandtl and Rayleigh numbers, respectively. k_0 is the thermal conductivity parameter. $\hat{\mathbf{g}} = \mathbf{g}/|\mathbf{g}|$ is the unified gravitational acceleration. Throughout this paper, vector valued functions are denoted by boldface.

The stationary Boussinesq equations (1)-(5) include, in addition to the velocity and the pressure fields, the temperature field, making it non-trivial to find the numerical solution. Early attempts on finding efficient numerical schemes to solve (1)-(5) were coupled finite element methods, such as the standard Galerkin finite element method [5], the low-order nonconforming finite element method [72], the least squared finite element method [57], the projection-based stabilized mixed finite element method [14], and the two-level finite element method [41]. These methods usually lead to coupled large systems to solve for $\mathbf{u}$, p, and θ simultaneously. Furthermore, the systems are also nonlinear and need iterations to handle the nonlinearity.

Exploiting the existing computing resources, various decoupled methods can reduce the computational cost by solving several smaller problems, be easily implemented based on the legacy code of the smaller problems, and speed up the computation by parallel computation, such as the iterative domain decomposition methods [6, 7, 9, 10, 11, 12, 22, 23, 24, 25, 28, 29, 30, 39, 53, 54, 55, 66, 79, 84], non-iterative domain decomposition methods [13, 18, 19, 26, 27, 36, 40, 65, 73, 104, 105], two-grid methods [2, 8, 60, 61, 82, 83], partition time-stepping methods [17, 62, 68], Lagrange multiplier methods [3, 34, 51, 103], explicit-implicit linearized stablization schemes [31, 32, 45, 58, 59, 71, 87, 94, 95, 102], the Invariant Energy Quadratization (IEQ) method [15, 81, 88, 90, 91, 92, 97, 99], the Scalar Auxiliary Variable (SAV) method [33, 52, 64, 69, 70, 98, 100], the zero-energy-contribution technique [85, 86, 87, 89, 96], and others [38, 46, 56, 67].

To avoid resulting large coupled systems, decoupled methods were also developed for the Boussinesq equations. By utilizing the data generated from previous iterative steps or temporal steps, the decoupled methods can decompose the original problem into several subsystems with smaller scales, and usually turn the original nonlinear problem into linearized ones. For stationary Boussinesq equations, the sequential iterative methods [42, 43] and two grid methods [74] are developed to decouple this problem. For the time-dependent case, an implicit-explicit (IMEX) scheme is proposed to decouple the system and solve the decoupled equations sequentially [93]. Extrapolation of velocities in previous temporal steps provides a prediction in the convection-diffusion equation, which decouples the whole system and linearizes the trilinear term in the convection-diffusion equation. Then the Navier-Stokes part is solved by using the solution obtained from the convection-diffusion equation.

In this paper, we aim to develop and analyze an efficient parallel iterative decoupling method for the stationary Boussinesq equations. The key technique is to design an iteration, which provides a convergent prediction for the coupling terms hence decouples the convection-diffusion equation from the Navier-Stokes equations. The decoupled subsystems do not have to wait for each other at each step of the iteration, thus can be solved in parallel. For the proposed method, we carry out the well-posedness, stability, and convergence analysis. Compared with the analysis in [5], a different mapping is introduced to prove the existence of the standard finite