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ANALYSIS OF WEAK GALERKIN FINITE ELEMENT METHODS WITH SUPERCLOSENESS

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Abstract. In [15], the computational performance of various weak Galerkin finite element methods in terms of stability, convergence, and supercloseness is explored and numerical results are listed in 31 tables. Some of the phenomena can be explained by the existing theoretical results and the others are to be explained. The main purpose of this paper is to provide a unified theoretical foundation to a class of WG schemes, where $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements are used for solving the second order elliptic equations (1)-(2) on a triangle grid in 2D. With this unified treatment, all of the existing results become special cases. The theoretical conclusions are corroborated by a number of numerical examples.

Key words. Weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, supercloseness, superconvergence.

1. Introduction

A weak Galerkin finite element method was presented by Wang and Ye in [12] to model the elliptic problems and then has been applied to solve various partial differential equations [1, 4, 5, 6, 7, 8, 9, 10, 11, 14, 18, 19].

The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding weak derivatives in algorithm design. Weak functions have the form of $v = \{v_0, v_b\}$, where v_0 and v_b can be approximated by polynomials in $P_\ell(T)$ and $P_s(e)$ respectively, where T stands for an element and e the edge or face of T, ℓ and s are non-negative integers. Weak gradients are defined for weak function in the sense of distributions and can be approximated in the polynomial space $[P_m(T)]^2$. Various combination of $(P_\ell(T), P_s(e), [P_m(T)]^2)$ leads to different weak Galerkin methods tailored for specific partial differential equations.

In [15], the computational performance of various weak Galerkin finite element methods in terms of stability, convergence, and supercloseness is explored and numerical results are listed in 31 tables. Some of the phenomena can be explained by the existing theoretical results and the others are to be explained. Table 1 (Table 6.3, [15]) shows the numerical results of a class of weak Galerkin schemes, where $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements are used for solving the second order elliptic equations (1)-(2) on a triangle grid in 2D. Some of the results in that table have theoretical explanations (such as elements 6.3.4, 6.3.8, and 6.3.12), while others are posed as open questions. The goal of this paper is to answer these open questions with a unified treatment. Furthermore, with this unified treatment, all of the existing results become spacial cases. As one of the main contributions of this paper, it is shown that by using $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements, the error between L^2-projection of the exact solution and the numerical solution will be dramatically reduced if the right parameter is used. More precisely, by choosing the appropriate

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TABLE 1 (TABLE 6.3, [15]).

Element $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ on triangular mesh, $|||\cdot||| = \mathcal{O}(h^{r_1})$ and $\|\cdot\| = \mathcal{O}(h^{r_2})$, t is defined in 7.

element	$P_k(T)$	$P_{k+1}(e)$	$[P_{k+1}(T)]^2$	t	r_1	r_2	Proved
6.3.1	$P_0(T)$	$P_1(e)$	$[P_1(T)]^2$	-1	0	0	N|N
6.3.2				0	1	1	N|N
6.3.3				1	2	2	N|N
6.3.4				∞	2	2	Y|Y
6.3.5	$P_1(T)$	$P_2(e)$	$[P_2(T)]^2$	-1	1	2	Y|N
6.3.6				0	2	3	N|N
6.3.7				1	3	4	N|N
6.3.8				∞	3	4	Y|Y
6.3.9	$P_2(T)$	$P_3(e)$	$[P_3(T)]^2$	-1	2	3	Y|N
6.3.10				0	3	4	N|N
6.3.11				1	4	5	N|N
6.3.12				∞	4	5	Y|Y

parameter, order one and two supercloseness for $k = 0$ and $k \geq 1$, respectively, can be obtained.

In this paper, we are concerned with the second order elliptic problem that seeks an unknown function u satisfying

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a polytopal domain in $\mathbb{R}^2$, ∇u denotes the gradient of the function u, and a is a symmetric 2×2 matrix-valued function in Ω. For simplicity, we shall assume that there exist two positive numbers $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \xi^t \xi \leq \xi^t a \xi \leq \lambda_2 \xi^t \xi, \qquad \forall \xi \in \mathbb{R}^2. \tag{3}$$

Here ξ is understood as a column vector and ξ^t is the transpose of ξ.

The paper is organized as follows. In Section 2, we shall describe a WG scheme for solving the second order elliptic equations (1)-(2). Section 3 is devoted to the discussion of the well posedness of the WG scheme. The error analysis for the WG solutions in an energy norm and in the L^2 norm will be investigated in Section 4 and Section 5, respectively. In Section 6, we shall present some numerical examples that confirm the theoretical estimates.

2. Weak Galerkin Finite Element Schemes

Suppose $\mathcal{T}_h$ is a quasi uniform triangular partition of Ω. For every element $T \in \mathcal{T}_h$, denote by h_T its diameter and $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{E}_h$ be the set of all the edges in $\mathcal{T}_h$.

First, we adopt the following notations,

$$\begin{aligned} (v, w)_{\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} (v, w)_T = \sum_{T\in\mathcal{T}_h} \int_T vw d\mathbf{x}, \\ \langle v, w\rangle_{\partial\mathcal{T}_h} &= \sum_{T\in\mathcal{T}_h} \langle v, w\rangle_{\partial T} = \sum_{T\in\mathcal{T}_h} \int_{\partial T} vw ds. \end{aligned}$$