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AN ARBITRARY-ORDER ULTRA-WEAK DISCONTINUOUS GALERKIN METHOD FOR TWO-DIMENSIONAL SEMILINEAR SECOND-ORDER ELLIPTIC PROBLEMS ON CARTESIAN GRIDS

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This paper is dedicated to my mother, Rebah Jetlaoui, who passed away from COVID in July 31, 2021 while I was completing this work.

Abstract. In this paper, we present and analyze a new ultra-weak discontinuous Galerkin (UWDG) finite element method for two-dimensional semilinear second-order elliptic problems on Cartesian grids. Unlike the traditional local discontinuous Galerkin (LDG) method, the proposed UWDG method can be applied without introducing any auxiliary variables or rewriting the original equation into a system of equations. The UWDG scheme is presented in details, including the definition of the numerical fluxes, which are necessary to obtain optimal error estimates. The proposed scheme can be made arbitrarily high-order accurate in two-dimensional space. The error estimates of the presented scheme are analyzed. The order of convergence is proved to be $p+1$ in the L^2-norm, when tensor product polynomials of degree at most p and grid size h are used. Several numerical examples are provided to confirm the theoretical results.

Key words. Ultra-weak discontinuous Galerkin method; elliptic problems; convergence; *a priori* error estimation.

1. Introduction

In this paper, we develop a new ultra-weak discontinuous Galerkin (UWDG) finite element method for the semilinear second-order elliptic problems of the form

$$-\Delta u + f(\mathbf{x}, u) = \mathbf{0}, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^d, \quad d = 1,\ 2,\ 3. \tag{1a}$$

We shall assume that the nonlinear function $f(\mathbf{x}, u) :\ \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is smooth with respect to its arguments $\mathbf{x}$ and u. To be more precise, we assume that f and its partial derivatives are continuous for $\mathbf{x} \in \overline{\Omega}$ and $u \in \mathbb{R}$ and satisfies the uniform bound

$$|f(\mathbf{x}, u)| \le M, \quad \forall\ \mathbf{x} \in \Omega, \quad \forall\ u \in \mathbb{R}, \tag{1b}$$

as well as the Lipschitz condition

$$|f_u(\mathbf{x}, u) - f_u(\mathbf{y}, v)| \le L\left(|\mathbf{x} - \mathbf{y}| + |u - v|\right), \quad \forall\ \mathbf{x},\ \mathbf{y} \in \Omega, \quad \forall\ u,\ v \in \mathbb{R}. \tag{1c}$$

For simplicity, we focus on two dimensions ($d = 2$) and write $\mathbf{x}$ as (x, y). In our analysis, we consider a rectangular domain denoted by $\Omega = \{\mathbf{x} = (x, y) :\ a < x < b,\ c < y < d\}$. We remark that our results remain true, with minor changes in the proofs, when the region Ω is a rectangular bounded domain of $\mathbb{R}^3$. In this paper,

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we will consider either periodic boundary conditions

$$\begin{aligned} &u(a,y)=u(b,y),\ u(x,c)=u(x,d),\\ &u_x(a,y)=u_x(b,y),\quad u_y(x,c)=u_y(x,d),\ \mathbf{x}\in\partial\Omega, \end{aligned} \tag{1d}$$

or mixed Dirichlet-Neumann boundary conditions

$$u=g_D,\quad \mathbf{x}\in\partial\Omega_D,\quad \mathbf{n}\cdot\nabla u=\mathbf{n}\cdot\mathbf{g}_N,\quad \mathbf{x}\in\partial\Omega_N, \tag{1e}$$

or purely Dirichlet boundary conditions

$$u=g_D,\quad \mathbf{x}\in\partial\Omega. \tag{1f}$$

Here, $\mathbf{n}$ is the outward unit normal to the boundary, $\partial\Omega$, of Ω. For the mixed boundary conditions (1e), we always assume that the boundary $\partial\Omega=\partial\Omega_D\cup\partial\Omega_N$ is decomposed into two disjoint sets $\partial\Omega_D$ and $\partial\Omega_N$ where Dirichlet and Neumann boundary conditions are imposed, respectively. We further assume that the measure of $\partial\Omega_D$ is nonzero. In our analysis, we assume that the given functions f, g_D, and $\mathbf{g}_N$ are smooth functions on their domains such that the problem (1) has one and only one solution $u\in H^2(\Omega)$. We refer the reader to [23, 25, 29] and references therein for the existence and uniqueness of solutions to elliptic problems.

The origin of the discontinuous Galerkin (DG) finite element method (FEM) can be traced back to [32, 34] where it has been introduced for discretizing the neutron transport equation. Since then various types of DG schemes have been successfully used to discretize differential equations containing higher order spatial derivatives. DG methods for elliptic problems have been introduced in the late 90's. They are by now well-understood and rigorously analyzed in the context of linear elliptic problems (cf. [5] for the Poisson problem). The most successful DG schemes include symmetric interior penalty DG (SIPG) methods, non-symmetric interior penalty DG (NIPG) methods, local DG (LDG) methods, direct DG (DDG) methods, and ultra-weak DG (UWDG) methods. The class of SIPG methods (introduced in [4, 35]) and the class of NIPG methods (considered in [14]) are important methods for higher order differential equations. Some of the general attractive features of these methods are the local and high order of approximation, the flexibility due to local mesh refinement and the ability to handle unstructured meshes and discontinuous coefficients. The SIPG and NIPG methods use penalties to enforce weakly both continuity of the solution and the boundary conditions. The LDG method was first introduced to solve general convection-diffusion problems by Cockburn and Shu [21]. Nowadays, the LDG method has been successfully used in solving many linear and nonlinear problems. The key idea of the LDG method is to first rewrite the equation with higher order derivatives into a first order system, then apply the standard DG method on the system by properly choosing the so-called numerical fluxes. The DDG method was first introduced by Liu and Yan [33]. It involves the interior penalty methodology since the scheme is based on the direct weak formulation. Unlike the LDG method, the DDG method is based on the direct weak formulation and the construct of the suitable numerical flux on the cell edges. This method is called DDG since it does not introduce any auxiliary variables in contrast to the LDG.

The class of UWDG methods are proposed in [18]. These methods are based on repeated integration by parts so that all spatial derivatives are shifted from the exact solution to the test function in the weak formulation. Unlike the LDG method, the UWDG method can be applied without introducing any auxiliary variables or rewriting the original equation into a larger system. In [18], Cheng and Shu