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ISOGEOMETRIC SOLUTION OF HELMHOLTZ EQUATION WITH DIRICHLET BOUNDARY CONDITION IN REGIONS WITH IRREGULAR BOUNDARY: NUMERICAL EXPERIENCES

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Abstract. In this paper we use the Isogeometric Analysis (IgA) to solve the Helmholtz equation with Dirichlet boundary condition over a bounded physical 2D domain. Starting from the variational formulation of the problem, we show how to apply IgA to obtain an approximated solution based on biquadratic B-spline functions. We focus the attention on problems where the physical domain has very irregular boundary. To solve these problems successfully a high quality parametrization of the domain must be constructed. This parametrization is also a biquadratic tensor product B-spline function, with control points computed as the vertices of a quadrilateral mesh with optimal geometric properties. We study experimentally the influence of the wave number and the parametrization of the physical domain in the accuracy of the approximated solution. A comparison with classical Finite Element Method is also included. The power of IgA is shown solving several difficult model problems, which are particular cases of the Helmholtz equation and where the solution has discontinuous gradient in some points, or it is highly oscillatory. For all model problems we explain how to select the knots of B-spline quadratic functions and how to insert knew knots in order to obtain good approximations. The results obtained with our implementation of the method prove that IgA approach is successful, even on regions with irregular boundary, since it is able to offer smooth solutions having at the same time some singular points and high number of oscillations.

Key words. Isogeometric analysis, Helmholtz equation, irregular regions.

1. Introduction

In its most general form Helmholtz equation in 2D is given by

$$-\Delta u(x,y) - k^2(x,y)u(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

where $k(x,y)$ and $f(x,y)$ are known functions and Δ denotes the Laplacian operator.

Due to its importance in different fields such as acoustic and electromagnetic systems, the case k equal to a positive constant have been intensively investigated over the years, see for instance [23], [24], [14], [29]. In this case, $u(x,y)$ is the amplitude of a wave traveling along Ω and k, called wave number, is the number of waves per unit of distance. The more complicated scenario, where $k(x,y)$ is a function depending on spatial variables, is also important. For instance, the wave function that satisfies a Schrödinger equation model of two interacting atoms [28] is solution of a Helmholtz equation with variable $k(x,y)$. Moreover, in realistic geophysical applications [16], $u(x,y)$ is the pressure wavefield and $k(x,y) = \omega^2/(\rho c^2)$, where ω is the angular frequency, $c = c(x,y)$ is the wave propagation velocity of the medium and $\rho = \rho(x,y)$ is the mass density, and both, $c(x,y)$ and $\rho(x,y)$ can contain high-contrast variations in space.

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If k is a small constant, Helmholtz equation can be solved successfully using low degree p *Finite Element Method* (FEM). But as k is increased, the number N_{FEM} of degrees of freedom necessary to obtain an accurate approximation of $u(x, y)$ must be proportional to k^2 [17], or the mesh size h have to be selected in such a way that $hk^{(p+1)/p}$ is constant and sufficiently small [27]. Moreover, many numerical difficulties appear [27], [14], [29], [17], [12]. These difficulties are associated with the fact that the standard variational formulation of the Helmholtz equation is sign-indefinite, hence for k sufficiently large, the coefficient matrix is indefinite and non-normal. As a consequence, iterative methods to solve the corresponding linear systems with N_{FEM} unknowns require the use of preconditioners, such as multigrid methods, domain decomposition, complex shifted Laplacian and deflation, see for instance [13], [32], [11].

Isogeometric analysis (IgA) was introduced by Hughes et al. in [21] as an extension of FEM to solve partial differential equations (PDE). The term *isogeometric* highlights that IgA uses B-splines functions twice: to parametrize Ω and as shape functions to approximate the solution of the PDE. In IgA approach B-splines functions may be constructed to have high smoothness. This is very important for problems with smooth solutions, where in comparison with FEM, IgA provides improved accuracy per degree of freedom. The first step to solve a PDE with IgA is the parametrization of Ω. This is currently an active research area, see for instance [34], [35], [30], [20], [36], [15], [31], [37] and [1]. The parametrization of the physical domain avoids the small errors introduced by FEM mesh, which can be amplified significantly in the context of wave propagation problems. In the last years IgA approach has been successfully used for a wide variety of PDE applications, see [3], [38], [6].

Previous work

Several recent papers [7], [9], [10], [12], adopted IgA as discretization technique to solve the Helmholtz equation with different boundary conditions. In [7] Helmholtz equation with Neumann boundary condition is solved in a domain described by 4 quadratic B-spline curves. Moreover, the acoustic in the interior of a simplified 2D model of a car is studied, for a wide frequency range. This problem is modeled with Helmholtz equation with Robin and Neumann boundary conditions. The results of the experiments in this paper confirm that, for similar degrees of freedom, IgA manage acoustic problems more efficiently than FEM, since it suffers less of the pollution errors, specially in the higher frequency range. In [9] partition of unity isogeometric analysis is applied for computing the scalar acoustic potential, governed by Helmholtz equation subject to (complex) Robin boundary condition. The study is focused in a comparison between FEM and IgA, including the use of enriched basis functions to reduce the pollution and also to avoid the need for domain re-meshing at high frequencies. The paper concludes that IgA has a clear advantage over FEM, since it achieves similar errors with significantly less degrees of freedom.

The numerical solution of the linear system derived from IgA discretization of Helmholtz equation is the central topic of papers [10] and [12]. In [10], Helmholtz equation is solved with Robin boundary condition. The performance of GMRES is investigated in the context of IgA, and it is compared to FEM, especially at high polynomial orders. The study includes the use of preconditioners such as ILU with a complex shift and complex shifted Laplacian. The conclusion is that to reach the convergence of GMRES, IgA needs fewer iterations compared to FEM. In [12],