INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 1, Pages 134–151

doi: 10.4208/ijnam2023-1006

THE MIXED FINITE VOLUME METHODS FOR STOKES PROBLEM BASED ON MINI ELEMENT PAIR

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Abstract. In this paper, we present and analyze MINI Mixed finite volume element methods (MINI-FVEM) for Stokes problem on triangular meshes. The trial spaces for velocity and pressure are chosen as MINI element pair, and the test spaces for velocity and pressure are taken as the piecewise constant function spaces on the respective dual grid. It is worth noting that the bilinear form derived from the gradient operator and the bilinear form derived from the divergence are unsymmetric. With the help of two new transformation operators, we establish the equivalence of bilinear forms for gradient operator between finite volume methods and finite element methods, and the equivalence of bilinear forms for divergence operator between finite volume methods and finite element methods, so the inf-sup conditions are obtained. By the element analysis methods, we give the positive definiteness of bilinear form for Laplacian operator. Based on the stability, convergence analysis of schemes are established. Numerical experiments are presented to illustrate the theoretical results.

Key words. Stokes problem, MINI element, mixed finite volume methods, inf-sup condition.

1. Introduction

The Stokes problem, as a basic problem in incompressible fluid mechanics and a classical prototype model of mixed problems, has been extensively studied. There are many research about finite element methods(MFEM), for which reader is referred to [19, 6, 7, 3, 28] and the references cited therein. The MFEM theory [4] shows that an elementary requirement, which makes discretization system corresponding to Stokes problems, is that the velocity-pressure element pair satifies the inf-sup condition. For solving the Stokes problem, there are many velocity-pressure element pairs to be constructed[1, 6], and many stabilization technology is proposed[5, 20]. The finite volume method is also a common numerical method for partial differential equations. Due to the local conservation property, fintie volume method is widely used in computational fluid dynamic[27, 16, 17, 2, 31, 18].

Finite volume element method(FVEM), which belongs to a kind of Petrov-Galerkin methods, is an important type of finite volume method. By choosing the Lagrange type finite element space as the trial space and using the piece constant function space on the dual grid as the test space, a complete theoretical framework of FVEM is established like finite element methods[27, 9, 42, 29, 37]. There are many scholars studied finite volume methods for Stokes problem. The finite volume methods by using the nonconforming elements space for velocity and the piecewise constants for pressure is studied in [11, 13, 39]. The finite volume methods by using the conforming elements space for velocity and the piecewise constants for pressure is studied in [12, 33, 41, 40]. Ye in paper [38] investigate the relatonshape between finie volume and finite element both conforming and unconforming velocity space and constants pressure space. The unified analysis and error estimation is established in [15, 26, 8]. For research on the finite volume method

Received by the editors on April 1, 2022 and, accepted on October 15, 2022.
2000 *Mathematics Subject Classification.* 65N08, 76D07, 76M12.
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whose velocity and pressure are both conforming elements space, they either use stablilized equal order pairs[22, 21, 32, 36], or discrete continuity equations by finte element methods[22, 32, 10]. MAC-like(Marker-And-Cell) finite volume methods on staggered grids are studied in the papers [12, 30, 35]. As the same time, scholars also extended the finite volume method to the Navier-Stokes equations[25, 23, 24] and other complex fluid problems[34].

In this article, we construct and analyze the MINI mixed finite volume element methods for Stokes problem. Based on the primary trianguler meshes, two different dual meshes associated with velocity and pressure are constructed. Then, the trial space for velocity and pressure are taken as MINI element pair, and the test spaces for velocity and pressure are choosen as the piecewise constants space on respective dual meshes. So we construct a full finite volume scheme for both momentum equation and continuity equation. Obviously, the schemes satisfy the local conservation of mass on dual element of pressure. However, the bilinear form derived from the gradient operator and the bilinear form derived from the divergence are unsymmetric, and they are all different from the corresponding bilinear form in mixed finite element methods. With the help of two new transformation operators, we establish the equivalence of bilinear forms for gradient operator between FVEM and MFEM, and the equivalence of bilinear forms for divergence operator between FVEM and MFEM, then the inf-sup conditions are obtained. Moreover, the equivalence of bilinear forms for gradient operator and divergence operator is obtained. The stability of bilinear form for Laplacian operator is proved by the element analysis methods. Based on the stability of saddle point system, the error estimations are proved.

The outline of the this paper is as follows. In section 2, we construct the MINI mixed finite volume methods for Stokes problem. In section 3, the continuity and stability of the bilinear forms are establish. We carry out the convergence analysis for the MINI mixed finite volume methods in section 4. In section 5, numerical experiments are presented to confirm the theoretical result.

2. The MINI mixed finite volume element methods

In this section, we establish the MINI mixed finite volume method for the Stokes equations

$$\left\{\begin{array}{rcll} -\nu\Delta\boldsymbol{u}+\nabla p &=& \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div}(\boldsymbol{u}) &=& 0, & \text{in } \Omega, \\ \boldsymbol{u} &=& \boldsymbol{0}, & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^2$, $\nu > 0$ is the kinematic viscosity, $\boldsymbol{u} = (u^x, u^y)$ is the fluid velocity, p is the pressure and $\boldsymbol{f} = (f^x, f^y)$ is the give body force per unit mass. For a non-negative integer k and $\mathcal{D} \in \mathbb{R}^2$, let $H^k(\mathcal{D})$ denote the Sobolev space with the norm $\|\cdot\|_{k,\mathcal{D}}$ and the semi-norm $|\cdot|_{k,\mathcal{D}}$. When $\mathcal{D} = \Omega$, we take $|\cdot|_k$ denote $\|\cdot\|_{k,\Omega}$. Then, we introduce the following spaces

$$L_0^2(\Omega) := \left\{p \in L^2(\Omega) : \iint_\Omega p dxdy = 0\right\}, \tag{2}$$

$$H_0^1(\Omega) := \left\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\right\}. \tag{3}$$