INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 1, Pages 29–46

doi: 10.4208/ijnam2023-1002

OPTIMAL CONTROL OF A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH HISTORY-DEPENDENT OPERATORS

YUJIE LI*, XIAOLIANG CHENG AND XILU WANG

Abstract. In this paper, we are concerned with an optimal control problem of a quasistatic frictional contact model with history-dependent operators. The contact boundary of the model is divided into two parts where different contact conditions are specified. For the contact problem, we first derive its weak formulation and prove the existence and uniqueness of the solution to the weak formulation. Then we give a priori estimate of the unique solution and prove a continuous dependence result for the solution map. Finally, an optimal control problem that contains boundary and initial condition controls is proposed, and the existence of optimal solutions to the control problem is established.

Key words. Variational inequality, contact problem, history-dependent operator, optimal control.

1. Introduction

Contact models play a significant role in mechanical engineering and have long been an important topic of research for scholars. The theory of variational inequalities [1, 2, 16, 17, 18] provides an effective way to study contact problems. As the research progresses, the concept of history-dependent operators was first introduced in [19]. These operators are used to model contact problems with long memory. The recent references related to history-denpendent operators can be found in [3, 11, 12, 13, 14, 20, 21, 29].

From the point of view of practical applications, it is great meaningful to study optimal control problems in contact mechanics. The subject of optimal control of variational inequalities was first studied in [23] and was developed by [24, 25, 26]. In [27], the existence of the solution to an optimal control problem is proved and the convergence for the regularized control problem is studied. The reference [6] and [7] prove the existence and approximation results of optimal solutions to a class of quasilinear elliptic variational inequalities and a nonlinear elliptic inclusion, respectively. In [28], the authors consider the numerical solutions for the optimal control of a class of variational-hemivariational inequalities and deduce the convergence result. As for evolutional case, the reference [14] studies an optimal control for a class of subdifferential evolution inclusions involving history-dependent operators and [4] focuses on the boundary optimal control of a dynamic frictional contact problem. The works of these two papers give us a great inspiration.

In this paper, we study an optimal control problem of a quasistatic friction contact problem involving history-dependent operators. The contact model was proposed in [8], and the special feature of the model lies on its contact boundary, which is divided into two parts with different contact conditions. The difference

Received by the editors on December 3, 2021 and, accepted on August 24, 2022.
2000 *Mathematics Subject Classification.* 47J20, 49J20, 74M15.
* Corresponding author: Yujie Li.

is that we consider boundary conditions related to the diaplacement field instead of the velocity field. In [8], the existence and uniqueness of the solution to the weak problem is proved and the error estimate for a discrete scheme is derived. However, the work of this paper is a useful exploration of the problem from another perspective. The main novelty is that we prove a continuous dependence result for the solution map of a quasistatic problem. Compared with dynamic problems, quasistatic problems [5, 10] are more difficult to derive a continuous dependence result and there is little relevant literature. Moreover, we consider control variables with regard to both boundary and initial conditions, and a cost functional that combines observations within the domain, on the boundary and at the terminal time. The techniques used in this work can also be applied to study some forms of variational-hemivariational inequalities, that is, weak formulations of some contact problems involving both convex and Clarke subdifferentials.

The rest of the paper is structured as follows. In Section 2 we recall some basic notation and present several preliminary results. In Section 3 we introduce a quasistatic contact problem with history-dependent operators. The existence and uniqueness of the solution is given and a priori estimate for the solution is proved. In Section 4, we deduce a continuous dependence result for the solution map based on the evolution inclusion. In Section 5, we prove that an optimal control problem has at least one solution, based on the continuous dependence result.

2. Notation and preliminaries

In this section, we recall some basic notation and known results that will be used later in the paper. Let X be a real Banach space. Throughout the paper, we denote by $\|\cdot\|_X$ and X^* the norm in X and its dual space, respectively. The notation X_w denotes X equipped with the weak topology. Furthermore, if X is a real Hilbert space, we denote by $(\cdot,\cdot)_X$ the inner product on X. We start with the definitions of the (convex) subdifferential and subgradient.

Definition 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Assume that $u \in X$ is such that $f(u) \neq \infty$. Then, the subdifferential of f at u is the set*

$$\partial f(x) = \{\xi \in X^* \mid f(v) - f(u) \geq \langle \xi, v-u \rangle_{X^* \times X}\,,\ \forall\, v \in X\}.$$

Each element $\xi \in \partial f(u)$ is called a subgradient of f at u.

For a function $\psi : X \to \mathbb{R} \cup \{+\infty\}$, we use the notation $D(\psi)$ for the effective domain of ψ, i.e.

$$D(\psi) = \{u \in X \mid \psi(u) \neq \infty\}.$$

The following lemma will be used in Section 3 to prove the unique weak solvability of a contact problem, and its proof can be found in [24], page 35.

Lemma 2.2. *Let X be a real Hilbert space and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper lower semicontinuous function. Then, for every $f \in L^2(0,T;X)$ and $u_0 \in D(\psi)$, there exists a unique function $u \in H^1(0,T;X)$ which satisfies*

$$u'(t) + \partial\psi(u(t)) \ni f(t) \quad a.e.\ t \in (0,T),$$
$$u(0) = u_0.$$

Then we recall the following consequence of the Banach fixed point theorem ([3], Lemma 3).