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# OPTIMAL BLOCK PRECONDITIONER FOR AN EFFICIENT NUMERICAL SOLUTION OF THE ELLIPTIC OPTIMAL CONTROL PROBLEMS USING GMRES SOLVER

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**Abstract.** Optimal control problems are a class of optimisation problems with partial differential equations as constraints. These problems arise in many application areas of science and engineering. The finite element method was used to transform the optimal control problems of an elliptic partial differential equation into a system of linear equations of saddle point form. The main focus of this paper is to characterise and exploit the structure of the coefficient matrix of the saddle point system to build an efficient numerical process. These systems are of large dimension, block, sparse, indefinite and ill conditioned. The numerical solution of saddle point problems is a computational task since well known numerical schemes perform poorly if they are not properly preconditioned. The main task of this paper is to construct a preconditioner the mimic the structure of the system coefficient matrix to accelerate the convergence of the generalised minimal residual method. Explicit expression of the eigenvalue and eigenvectors for the preconditioned matrix are derived. The main outcome is to achieve optimal convergence results in a small number of iterations with respect to the decreasing mesh size $h$ and the changes in $\delta$ the regularisation problem parameters. The numerical results demonstrate the effectiveness and performance of the proposed preconditioner compared to the other existing preconditioners and confirm theoretical results.

**Key words.** Partial differential equations (PDEs), PDE-optimal control problems, saddle point problem, block preconditioners, preconditioned generalised minimal residual method (PGMRES).

## 1. Introduction

Optimal control problems associated with partial differential equations arise in a variety application areas such as social, scientific, industrial, medical and engineering applications including optimal control, optimal design and parameter identification. In particular real life applications include flow control, reaction-diffusion problem of chemical processes, shape optimization, problems in financial markets and optimal pricing. In this paper we deal with the numerical solution of the distributed optimal control of elliptic equations that arise in real life applications in the optimal stationary heat. We consider the following elliptic distributed PDE-optimal control problem

$$\min_{(\mathbf{u},\mathbf{y})} J(\mathbf{y},\mathbf{u}) := \frac{1}{2} \parallel \mathbf{y}-\mathbf{y}_d \parallel^2_{L^2(\Omega)} + \frac{\delta}{2} \parallel \mathbf{u} \parallel^2_{L^2(\Omega)}, \tag{1}$$

subject to the constraints

$$\begin{aligned} -\Delta \mathbf{y} &= \mathbf{f}+\mathbf{u} \quad \text{in} \quad \Omega, \\ \mathbf{y} &= \mathbf{g} \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

where $\Omega \subset \mathcal{R}^2$ is the domain with boundary $\partial\Omega$. These problems were theoretically introduced by [11, 22] comprise of the objective function given by Equation (1) to

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be minimised and the PDE-constraints given by Equations (2). Here we want to find $\mathbf{y}$ the state variable that satisfies the PDE-constraint as close to $\mathbf{y}_d$ as possible, the desired state which is known over the domain $\bar{\Omega}$ and $\mathbf{u}$ the control variable on the right hand side. This means that $\mathbf{y}$ is the solution of Equation (2) for a given control $\mathbf{u}$ either in the whole domain or on the boundary. The control functions that are either distributed (defined on $\Omega$) or boundary (defined on $\partial\Omega$). If control functions are defined on $\partial\Omega$ we have boundary control problem for example the optimal temperature distribution otherwise we have a distributed control problem like the optimal heat source distributed on the whole domain. The temperature distribution or state $\mathbf{y}$ inside the domain is controlled by the enforcing heating source $\mathbf{u}$. For some practical purposes, we would like to choose the optimal control which minimizes the difference between the desired stationary temperature distribution $\mathbf{y}_d$ and the achievable temperature distribution $\mathbf{y}$. Mathematically, by assuming the boundary temperature vanishes. In this paper, we develop a new fast and efficient solver for the distributed optimal control problem Equations (1-2). The parameter $\delta$ is called the regularization parameter which measures the cost of the control and is supplied and positive. We refer to [3, 8, 9] on their numerical developments of such problems.

The optimal control problem has a unique solution $(\mathbf{y}, \mathbf{u})$ characterised by the optimality system called the Karush-Kuhn-Tucker (KKT) system [4, 20]. The first order optimality system of the PDE-optimal control problem Equations (1-2) consists state equation, adjoint equation and the control equation which is a saddle point problem as given below

$$\begin{aligned} &(3) \quad & -\Delta\mathbf{p} &= \mathbf{y}-\mathbf{y}_d, \quad \text{in } \Omega \quad & \mathbf{p}=\mathbf{0} \quad \text{on } \partial\Omega \quad & \text{adjoint equation,}\\ &(4) \quad & -\Delta\mathbf{y} &= \mathbf{f}+\mathbf{u}, \quad \text{in } \Omega \quad & \mathbf{y}=\mathbf{g} \quad \text{on } \partial\Omega \quad & \text{state equation,}\\ &(5) \quad & \delta\mathbf{u}-\mathbf{p} &= 0 \quad \text{in } \Omega \quad & & \text{control equation.} \end{aligned}$$

The optimality system is achieved through the Lagrange multiplier method which partitions the model problem into three equations namely in the state $\mathbf{y}$, control $\mathbf{u}$ and the adjoint, $\mathbf{p}$. For the numerical solution of the elliptic optimal control problem we apply the finite element method to the Equations (3 - 5) to get the linear saddle point problem. The finite element method is the most popular technique for the numerical solution of the PDE-constrained optimisation problems, see [9, 17, 18]. The finite element method results in the coupled linear algebraic system which has to be solved by the appropriate solvers. The resulting discrete KKT system is

$$(6) \qquad \mathcal{K}\mathbf{x} = \begin{pmatrix} M & O & K \\ O & \delta M & -M \\ K & -M & O \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} M\mathbf{y}_d \\ \mathbf{0} \\ \mathbf{d} \end{pmatrix} = \mathbf{b},$$

where $K \in \mathcal{R}^{n\times n}$ is a stiffness matrix and mass matrix $M \in \mathcal{R}^{n\times n}$ both symmetric and positive definite. Both $K$ and $M$ are sparse, hence $\mathcal{K}$ is sparse, symmetric and indefinite. The vector $\mathbf{d} \in \mathcal{R}^n$ contains the terms arising from the boundaries of the finite element of the state $\mathbf{y}$.

The linear algebraic system of Equation (6) is large scale, indefinite and has poor spectral properties such that well known Krylov subspace methods perform poorly [3, 18] and references therein. In recent years, the efficient solvers of the algebraic system that results from the optimal control problems has attracted a lot of attention and plenty of algorithms and preconditioners are proposed. The vital requirement for optimal performance of the Krylov subspace iterative methods is that the system matrix must have good spectral properties. This has preoccupied