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A SPRING–BEAM SYSTEM WITH SIGNORINI'S CONDITION AND THE NORMAL COMPLIANCE CONDITION

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Abstract. This paper provides mathematical and numerical analyses for a dynamic frictionless contact problem in which both of Signorini's condition and the normal compliance condition are used. The contact problem is considered by employing two viscoelastic (Kelvin-Voigt type) objects: a linear Timoshenko beam and a nonlinear spring. In addition, a transmission condition is imposed on one end of the beam and the top of the spring so that they can touch and vibrate together. We prove the existence of solutions satisfying all the conditions. Time discretizations and finite element methods are utilized to propose the fully discrete numerical schemes. We select several groups of data to present and discuss numerical simulations.

Key words. Timoshenko beams, duffing equation, normal compliance, Signorini's condition, Galerkin's method, contraction mapping argument.

1. Introduction

Over the past half century, mathematical theories and numerical methods of contact mechanics have made remarkable progress. Reader may refer to the monographs [6, 7] to understand mathematical and numerical approaches to contact problems with various effects such as friction, wear, thermal effects, adhesion, or damage. In particular, one–dimensional dynamic contact problems for strings, rods, or beams with or without those side effects have been actively studied. See, e.g., [21, 23, 19, 15, 20, 16, 9, 17, 18]. If they are integrated, more practical contact models can be built to describe a wide range of physical or engineering situations. Recently, in [22, 28], a rod–beam system is constructed to provide mathematical and numerical analyses on a V–shaped Micro-Electro-Mechanical Systems (MEMS) actuator.

In this work, we utilize two Kelvin–Voigt typed viscoelastic objects with the two contact conditions and a transmission condition to design a mechanical system. A nonlinear spring and a linear beam coalesce into a dynamic contact model to consider a spring–beam system which results in a system of differential equations with the three conditions aforementioned. This contact model can be regarded as a boundary thin deformable obstacle problem. The system consists of an initial–boundary value problem (IBVP) and an extended Duffing equation (see the book [3] for the original Duffing equation) with initial data. The Duffing equation written by a second order nonlinear ordinary differential equation (ODE) describes the motion of a viscoelastic spring. The IBVP which contains a couple of linear partial differential equations (PDEs) also describes the motion of a viscoelastic Timoshenko beam [11, 12]. Unlike most contact models, our contact model is considered by using both the normal compliance condition (see e.g., [13, 14] and references therein) and Signorini's condition. The first condition is for between one end of the beam and the top of the spring and the other is for when the spring is fully compressed and hits a rigid foundation. One can notice that the normal compliance is a regularization of

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contact forces in Signorini's condition. A similar but simpler contact model can be found in [10]. In addition to the two contact conditions, a transmission condition is combined with the normal compliance condition, when the end of the beam and the top of the spring touch each other. In fact, the transmission condition is inspired by the model of a beam on Emil Winkler's elastic foundation, [2] published in 1867. His model has received special attention from civil engineering communities, since it has simple but profound assumptions. We note that the springs used in the Winkler's model are linearly elastic and that there are multiple contacts between a beam and the springs.

In many papers (e.g., [25, 26, 27, 24] and references therein), the Duffing equation is first modified and then the analytical solutions are found. In future work, the first author in this paper will generalize the Duffing equation more mathematically and then add dynamic frictional contact conditions formed by Coulomb's laws (see [8]). Due to the complexity of the problem, the solutions are unlikely to be found. Instead, its solvability will be studied and numerical schemes will be proposed.

In order to prove the existence results for the contact problem, we derive a corresponding variational formulation to the IBVP in the abstract setting and then apply Galerkin's method to it. Signorini's condition essentially causes nonsmooth solutions but the convolution of the standard mollifier and locally integrable solutions is employed to regularize them. While contact forces understood at the atomic level (see [1, Chapter I]) do not guarantee the conservation of energy or energy balance with viscosity, the regularized contact forces by means of the standard mollifier are likely to do so. Such a discrepancy will be explored in future work. We note that the regularization of the contact forces would be unrelated to the normal compliance condition.

We use a contraction mapping argument to prove the uniqueness of the solutions satisfying all the conditions except for Signorini's condition. We note that it still remains open to prove the uniqueness for Signorini's condition in the dynamic case.

The fully discrete numerical schemes are proposed, based on time discretizations on a time interval and finite element methods (FEMs) in the spacial domain. The overall scheme for computing each time step numerical approximations is to establish recursive relations and employ the the Newton–Raphson method. Particularly, block matrix manipulations and back substitutions are required to obtain the fully discrete numerical approximations of the PDEs. The Algorithm 1 presented in the section 5 explains all detailed steps. We also investigate numerical stability which is supported by numerical results, as we shall see them later.

This paper is organized as follows. The detailed illustrations for the mathematical model are presented in Section 2 and mathematical background is introduced in Section 3. In Section 4, we prove the existence of the global solutions in the continuous case. In Section 5, the fully discrete numerical schemes are explained and an algorithm is provided to compute numerical solutions. A criteria for numerical stability is also formed and validated. Several groups of data are chosen and then numerical simulations and results are presented and discussed in the last section 6.

2. A mathematical model

See Figure 1 to understand the dynamic contact model. An easiest way to understand the model is to consider the motion of seesaws at playgrounds. For a mathematical reason, we want to consider the motion of *a half* board which is described by a coupled PDEs (1–2). The half of the board is assume to be a linear