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NUMERICAL ANALYSIS OF A STRUCTURE-PRESERVING SPACE-DISCRETIZATION FOR AN ANISOTROPIC AND HETEROGENEOUS BOUNDARY CONTROLLED N-DIMENSIONAL WAVE EQUATION AS A PORT-HAMILTONIAN SYSTEM

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Abstract. The anisotropic and heterogeneous N-dimensional wave equation, controlled and observed at the boundary, is considered as a port-Hamiltonian system. A recent structure-preserving mixed Galerkin method is applied, leading directly to a finite-dimensional port-Hamiltonian system: its numerical analysis is carried out in a general framework. Optimal choices of mixed finite elements are then proved to reach the best trade-off between the convergence rate and the number of degrees of freedom for the state error. Exta compatibility conditions are identified for the Hamiltonian error to be twice that of the state error, and numerical evidence is provided that some combinations of finite element families meet these conditions. Numerical simulations are performed in 2D to illustrate the main theorems among several choices of classical finite element families. Several test cases are provided, including non-convex domain, anisotropic or heterogeneous cases and absorbing boundary conditions.

Key words. Port-Hamiltonian systems, N-dimensional wave equation, finite element method, structure-preserving discretization, numerical analysis.

1. Introduction

The present work addresses the numerical analysis of a structure-preserving space-discretization of an N-dimensional wave equation with boundary control in the formalism of port-Hamiltonian systems. Since it is intended to merge several points of view on the same subject, the authors have taken care to be pedagogical in each section, hence trying to talk to several scientific communities. This choice of presentation will certainly lead readers to find some parts related to his/her domain(s) of research unnecessary. Roughly speaking, this paper is intended to port-Hamiltonian specialists, numerical analysts, and scientific computing users.

1.1. Port-Hamiltonian systems.

In the last two decades, infinite-dimensional port-Hamiltonian systems (pHs) [57, 46] have proved to be a very accurate way to model and control complex multi-physics open systems. This framework enjoys several advantages, such as a relevant physical meaning and a useful underlying geometrical structure (namely Stokes-Dirac structure). It has to be pointed out that, even if known Partial Differential Equations (PDEs) are often *only rewritten* in the pHs formalism in general, this powerful tool also allows a direct modelling of physical systems (see for instance [17, 48, 3]) which proves useful to derive PDEs. Furthermore, it is intrinsically modular: interaction systems (such as fluid-structure interactions [14], heat-wave interactions [32], plasma in a tokamak [58], etc.) can be described through the interconnection of several subsystems with a port-Hamiltonian structure, leading to a more complex pHs [15, 40]. It finally leads to a power balance, expressing the variation of the Hamiltonian functional (often

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chosen as the system total energy), especially *via* boundary controls and boundary observations.

1.2. Structure-preserving discretisation. A recent topic of research is to provide accurate (space-) discretization methods to preserve this powerful formalism. Roughly speaking, mainly two non-exclusive communities work on the issue of *structure-preserving discretization*. The first one makes use of exterior calculus, while the other makes use of vector calculus. It is known that the two points of view are well-founded, and several strategies to merge their advantages efficiently have already been proposed for several discretization issues (see *e.g.* [36] and the many references therein).

In the present work, a method for the preservation of the power balance of the Hamiltonian (encoded in an underlying Stokes-Dirac structure) is studied. In the wide literature, several strategies have been proposed: we can cite *e.g.* [33, 52, 38] for geometric discretizations, [41, 18] for Galerkin methods and [54] for finite differences method. However, some of these strategies seem difficult to carry over to N-dimensional systems or to apply to complex geometries, while others require post-processing to construct the finite-dimensional Dirac structure. Another structure-preserving community works on the preservation of the de Rham cohomology and related decompositions (such as the Hodge-Helmholtz decomposition). This topic is older and finds its origins in problems such as electromagnetism (see *e.g.* [44] and references therein). It is often written in the exterior calculus formalism, allowing for more abstraction, hence more generality, for the construction of discrete differential operators: see *e.g.* [34, 9, 5, 6, 27] for theoretical aspects, and [22, 21, 28] for some applications to partial differential equations.

According to these definitions of structure-preserving discretization, a numerical method for port-Hamiltonian systems should be able to take into account the aforementioned continuous properties at the discrete level. Indeed, this would lead to a relevant physical meaning for the computed quantities (without post-processing), together with an obvious manner to distribute the computations thanks to the modularity property: in particular, each sub-system could be reduced through a structure-preserving model reduction [30, 19, 31] *prior to* their interconnections. Furthermore, in the field of automatic control, several methodologies for efficient control or stabilisation rely on the pHs form of the approximate finite-dimensional system [53]: this encourages research for efficient structure-preserving methods of infinite-dimensional pHs, and even more those related to boundary-controlled-and-observed PDEs.

A special case of the mixed Galerkin method, called the Partitioned Finite Element Method (PFEM) [13], seems to be one of the most adapted scheme to build a mimetic finite-dimensional Dirac structure [46].

In [35], the numerical method proposed for the spatial discretization of *closed* hyperbolic systems, based on the *primal-dual* or *dual-primal formulations* given in [35, Eqs. (15) and (16)], and making use of an abstract mixed Galerkin method, can be seen as the starting point of the PFEM for *closed* systems. Indeed, the idea of *partitioning* the system to choose on which equation an integration by parts should be applied was already mentioned: "the principle is to multiply the two equations [...] by test functions and to integrate over Ω, but the key point this time is to apply integration by parts only for one of the two equations.", see [35, p. 207]. The new difficulty for port-Hamiltonian systems lies in the boundary terms, namely the control and the observation. The present work investigates the issue of accurate approximations at the boundary.