INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 2, Pages 176–198

doi: 10.4208/ijnam2023-1008

IMPROVED LONG TIME ACCURACY FOR PROJECTION METHODS FOR NAVIER-STOKES EQUATIONS USING EMAC FORMULATION

SEAN INGIMARSON, MONIKA NEDA, LEO G. REBHOLZ, JORGE REYES, AND AN VU

Abstract. We consider a pressure correction temporal discretization for the incompressible Navier-Stokes equations in EMAC form. We prove stability and error estimates for the case of mixed finite element spatial discretization, and in particular that the Gronwall constant's exponential dependence on the Reynolds number is removed (for sufficiently smooth true solutions) or at least significantly reduced compared to the commonly used skew-symmetric formulation. We also show the method preserves momentum and angular momentum, and while it does not preserve energy it does admit an energy inequality. Several numerical tests show the advantages EMAC can have over other commonly used formulations of the nonlinearity. Additionally, we discuss extensions of the results to the usual Crank-Nicolson temporal discretization.

Key words. Navier-Stokes equations, EMAC formulation, projection methods.

1. Introduction

It is widely accepted that the Navier-Stokes equations (NSE) determine the evolution of incompressible, viscous, Newtonian fluid flow. These equations are given by

$$u_t + u \cdot \nabla u + \nabla p - \nu \Delta u = f, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

where u and p represent velocity and pressure respectively, f is an external force, and ν represents kinematic viscosity, which is inversely proportional to the Reynolds number Re. Appropriate boundary and initial conditions are needed to close the system.

While the NSE are built from conservation of linear momentum and mass conservation, they are also well-known to conserve energy, angular momentum, enstrophy in 2D, helicity in 3D, among other important physical quantities [10]. By 'conserve' we refer to the case of no viscous or external forces, but if these forces are present than an exact balance can be derived where the nonlinearity plays no role. In addition to being conserved, these quantities are believed to play a critical role in flow structure development, the energy cascade and energy dissipation, and the microscale [8, 10, 27]. However, in most NSE simulations, very few or none of these quantities are exactly conserved [3, 4]. Often, energy is at least bounded, as this is required for numerical stability. However, in most finite element computations mass is only weakly conserved [16], and this in turn breaks the conservation of momentum, angular momentum and other important physical quantities [3]. One solution to this problem is to use strongly divergence-free discretizations, such as Scott-Vogelius finite elements, however these elements can require mesh restrictions and higher degree polynomials, especially in the case of quadrilateral elements.

Received by the editors on April 26, 2022; accepted on November 10, 2022.
2000 *Mathematics Subject Classification.* 65M60, 76D05, 35Q30.

Another approach is to change the form of the nonlinearity to the EMAC (Energy, Momentum, and Angular momentum Conserving) form proposed in [3], where the identity

$$u \cdot \nabla u + \nabla p = 2D(u)u + (\nabla \cdot u)u + \nabla P,$$

was derived, with $P = p - \frac{1}{2}|u|^2$. There it was shown that the NSE with this nonlinear formulation used in (1)-(2) and discretized with standard elements such as Taylor-Hood or the mini element, conserves energy, momentum and angular momentum, as well as particular definitions of 2D enstrophy and 3D helicity. This is in contrast to the more commonly used rotational, skew-symmetric, convective and conservative forms, none of which conserve all of energy, momentum and angular momentum [3].

Since the original EMAC paper [3] in 2017, EMAC has garnered a considerable amount of attention in the CFD community. It has been used in problems involving vortex-induced vibration [24], turbulent flow simulation [18], noise radiated by an open cavity [21], high Reynolds number vortex dynamics [30], and more [7, 6, 23, 19, 2]. These numerical results have all been quite favorable, but there is still much to be done for its analytical study. What is proven so far is results for conservation properties [4], stability and convergence [4], efficient algorithms and linearization development [4], and a longer time accuracy result that shows the Gronwall exponent from EMAC is independent of the viscosity [22].

The purpose of this paper is to extend the study of EMAC to the case of a projection method temporal discretizations together with finite element spatial discretization. Projection methods were originally developed by Temam [32] and Chorin [5], and work using a Hodge type decomposition idea to split the NSE into two steps: the first solves the momentum equation without a divergence-free constraint, and the second projects the step 1 solution into the divergence free space. There have been many improvements made to projection methods over the years[1], but they all are still based on the fundamental decomposition / splitting from the original development. Analysis of projection methods is rather different and more complex than for standard coupled schemes, see e.g. [11, 25, 31], and herein we will extend the study of EMAC discretizations using to projection methods.

This paper is organized as follows. In section 2, we provide mathematical notation and preliminary information for the analysis. In section 3 we introduce the projection method algorithm and show the conservation properties of it. Stability and error analysis are presented in section 4. Section 5 further extends out work to coupled schemes for both EMAC and SKEW. Numerical tests can be found in section 6 followed by concluding remarks in the last section 7.

2. Notation and Preliminaries

We present in this section the necessary notation and mathematical preliminaries for a smooth analysis to follow. We assume a convex polygonal (or smooth boundary) domain $\Omega \subseteq \mathbb{R}^d$ where $d = 2, 3$. The $L^2(\Omega)$ inner product is denoted as $(\cdot, \cdot)$ and the $L^2(\Omega)$ norm with $||\cdot||$. Other norms will be clearly labeled with subscripts.

[1]The folklore, as told to author LR by a former Chorin student, is that for many years no Chorin student was allowed to graduate without improving on projection methods.