INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 4, Pages 478–496

doi: 10.4208/ijnam2023-1020

WELL-POSEDNESS AND CONVERGENCE ANALYSIS OF A NONLOCAL MODEL WITH SINGULAR MATRIX KERNEL

MENGNA YANG AND YUFENG NIE*

Abstract. In this paper, we consider a two-dimensional linear nonlocal model involving a singular matrix kernel. For the initial value problem, we first give well-posedness results and energy conservation via Fourier transform. Meanwhile, we also discuss the corresponding Dirichlet-type nonlocal boundary value problems in the cases of both positive and semi-positive definite kernels, where the core is the coercivity of bilinear forms. In addition, in the limit of vanishing nonlocality, the solution of the nonlocal model is seen to converge to a solution of its classical elasticity local model provided that $c_t = 0$.

Key words. Nonlocal model, well-posedness, convergence, singular matrix kernel, coercivity.

1. Introduction

In this paper, we consider a two-parameter nonlocal model as follows,

$$\mathbf{u}_{tt}(t,\mathbf{x}) = \mathcal{L}_\delta \mathbf{u}(t,\mathbf{x}) + \boldsymbol{b}(t,\mathbf{x}),\ (t,\mathbf{x}) \in (0,T)\times\mathcal{S}, \tag{1}$$

where the nonlocal integral operator $\mathcal{L}_\delta$ is given by

$$\begin{aligned}\mathcal{L}_\delta \mathbf{u}(t,\mathbf{x}) := \int_{\mathcal{S}\cup\Omega_\delta} &\left(\frac{c_n(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} + \frac{c_t\big[(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})\big]^*}{|\mathbf{x}'-\mathbf{x}|^2}\right)\\ &\times \big(\mathbf{u}(t,\mathbf{x}') - \mathbf{u}(t,\mathbf{x})\big)\chi_\delta(\mathbf{x}'-\mathbf{x})d\mathbf{x}',\end{aligned} \tag{2}$$

$\mathcal{S} \subseteq \mathbb{R}^2$ is an open domain ($\mathcal{S} = \Omega$ or $\mathcal{S} = \mathbb{R}^2$), $\Omega_\delta = \{\mathbf{x} \in \mathbb{R}^2 \setminus \Omega : \mathrm{dist}(\mathbf{x}, \partial\Omega) \le \delta\}$ is a collar domain surrounding a bounded open set $\Omega \subseteq \mathbb{R}^2$. $\mathbf{u} : (0,T)\times\mathcal{S}\cup\Omega_\delta$ represents displacement, and $\boldsymbol{b}$ is the external force density. c_n, c_t denote the tensile parameter and shear parameter, their expressions can be derived as

$$c_n = \frac{8E(1+\nu)}{\pi\delta^4(1-\nu^2)},\ c_t = \frac{8E(1-3\nu)}{\pi\delta^4(1-\nu^2)}, \tag{3}$$

here E, ν are the Young's modulus and Poisson's ratio, and we note that $0 \le \nu \le 1/3$. The horizon parameter δ characterizes the effective range of nonlocal interaction between the material point $\mathbf{x}'$ and point $\mathbf{x}$, and $\chi_\delta(\cdot)$ is the standard canonical function, i.e.,

$$\chi_\delta(\mathbf{x}'-\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}'-\mathbf{x}| \le \delta,\\ 0, & \text{otherwise.}\end{cases}$$

In recent years, there have been lots of works done on nonlocal equation of the type (1) and relevant variational problems, including theory analysis [1, 2, 3], numerical methods [4, 5, 6, 23], model development and applications [8, 20, 29]. Regarding the well-posedness theory for equations similar to (1), Emmrich and Weckner [13, 14] proved the well-posedness of the initial problem on bounded domains by using

---

Received by the editors on December 6, 2022 and, accepted on March 13, 2023.
2000 *Mathematics Subject Classification.* 45F15, 45K05, 46E40, 74B99.
*Corresponding author.

semigroup theory of operators. In [17], the well-posedness of a scalar nonlocal evolution problem is obtained by utilizing properties of Neumann series and Volterra integral equations, where the boundary data is proposed on the classical boundary domain $\partial\Omega$. In particular, Du and Zhou [21] established the well-posedness results for a nonlocal initial problem in the Fourier space, which takes into account the non-integrable kernels. In addition, Aksoylu and Parks [15] considered scalar linear stationary nonlocal problems, and gave the well-posedness results, the key step is to utilize domain decomposition methods to prove the coercivity, (see also [16] for a similar discussion). More generally, Mengesha and Du [12] proved the well-posedness for a nonlinear stationary nonlocal problem based on variational methods. We refer to [10, 17, 18] for an exhaustive introduction of well-posedness results.

On the other hand, observe that δ acts as a bridge between nonlocal models and the corresponding local models, so the study of reduction of nonlocal models to local models in the limit of $\delta \to 0$ has attracted much attention. In [10], the authors proved that the nonlocal integral operator applied to smooth functions converges asymptotically to the corresponding classical differential operator by using using Taylor expansion. In particular, based on Fourier transform, Mikata [11] analyzed the limit behaviors of solutions for a kinds of peristatic and peridynamic nonlocal problems, where solutions of these nonlocal equations approach solutions of the corresponding local equations with horizon vanishes. More results can be found in [7, 9, 19, 21] and references therein.

Inspired by the above papers [10, 15, 21], we will prove the well-posedness and convergence results for the initial and stationary cases of equation (1), which are the focus of our paper. For the well-posedness results of stationary nonlocal problems, the coercivity of bilinear forms is ensured by using relative compactness and some key inequalities. In particular, we don't rely on the proof in [10, 21] for convergence results as $\delta \to 0$, but made some modifications, and introduce some other techniques.

This paper is organized as follows. In Section 2, for the initial value problem associated to equation (1), we prove the well-posedness results and energy conservation via Fourier transform. In Section 3, for the corresponding Dirichlet-type nonlocal boundary problems, the well-posedness results of solution are established in the cases of positive definite kernel and semi-positive definite kernel. In Section 4, we shall analyze the limit behaviors of solutions of nonlocal problems as $\delta \to 0$. Finally, we complete the paper with an appendix.

Notation 1.1. Throughout the paper, we will denote various generic positive constants by the same letter C, although the constants may differ from line to line. Moreover, relevant dependencies on parameters will be emphasized using parentheses, i.e., $C \equiv C(T, \delta)$ means that C depends only on T, δ. The notation $\otimes$ denotes the dyadic product and $[(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})]^* = \mathbb{I}-(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})$, here $\mathbb{I}$ is a second order identity matrix. $(\cdot,\cdot)$ is the inner product defined as $(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^2} \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d\mathbf{x}$. M is a finite number.

To achieve our main results, let us first give a brief review of model equation (1). Equation (1) can be deduced from the following two-parameter nonlocal peridynamic model by a series of simplification,

$$\mathbf{u}_{tt}(t,\mathbf{x}) = \int_{B_\delta(\mathbf{x})} (c_n\eta_n\hat{\mathbf{e}}_n + c_t\eta_t\hat{\mathbf{e}}_t)d\mathbf{x}' + \boldsymbol{b}(t,\mathbf{x}), (t,\mathbf{x}) \in (0,T)\times\mathcal{S}. \tag{4}$$