INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 4, Pages 538–556

doi: 10.4208/ijnam2023-1023

HIGH ORDER METHOD FOR VARIABLE COEFFICIENT INTEGRO-DIFFERENTIAL EQUATIONS AND INEQUALITIES ARISING IN OPTION PRICING

PRADEEP KUMAR SAHU AND KULDIP SINGH PATEL*

Abstract. In this article, the implicit-explicit (IMEX) compact schemes are proposed to solve the partial integro-differential equations (PIDEs), and the linear complementarity problems (LCPs) arising in option pricing. A diagonally dominant tri-diagonal system of linear equations is achieved for a fully discrete problem by eliminating the second derivative approximation using the variable itself and its first derivative approximation. The stability of the fully discrete problem is proved using Schur polynomial approach. Moreover, the problem's initial condition is smoothed to ensure the fourth-order convergence of the proposed IMEX compact schemes. Numerical illustrations for solving the PIDEs and the LCPs with constant and variable coefficients are presented. For each case, obtained results are compared with the IMEX finite difference scheme, and it is observed that proposed approach significantly outperforms the finite difference scheme.

Key words. Schur polynomials, implicit-explicit schemes, partial integro-differential equations, jump-diffusion models, option pricing.

1. Introduction

The assumptions of log-normal distribution of underlying assets and constant volatility considered by Black and Scholes [1] to derive the partial differential equation (PDE) have been proven inconsistent with the real market scenario. Consequently, the research community came up with advanced models to elaborate the term like negative skewness, heavy tails, and volatility smile. In one of those efforts, the jumps phenomenon was incorporated into the dynamics of the underlying asset by Merton [2] to surpass the shortcomings of the Black-Scholes model, and the model is termed as Merton's jump-diffusion model. The PIDEs for pricing European options and the LCPs for pricing American options were obtained under Merton's jump-diffusion model. Since the analytical solution for these PIDEs and LCPs does not exist in general, it is inevitable to apply numerical methods to solve these equations.

Let us now briefly review the existing finite difference based numerical methods for solving PIDEs and LCPs. An IMEX finite difference method (FDM) has been developed in [3] for pricing European and Barrier options. The convergence of the proposed IMEX method has also been proved. In [4], a fully implicit FDM has been proposed for solving the PIDEs, and the stability of the method has also been proved. Three time levels implicit FDM were proposed for pricing European and American options in [5] and [6] respectively. All these numerical methods are atmost second order accurate, and it is one of those concerns where further research is required.

Received by the editors on on September 19, 2022 and, accepted on April 17, 2023.
2000 *Mathematics Subject Classification.* 65M06, 65M12, 91G20.
*Corresponding author.

It is observed that substantial increment in the number of grid points of computational stencil may result in high-order accurate FDM, however the implementation of boundary conditions would become tedious in such a case. Moreover in such a case, the discretization matrices with more bandwidth appears in fully discrete problem and it may also suffer from restrictive stability conditions. Therefore, FDMs have been developed using compact stencils at the expense of some complication in their evaluation and these are commonly known as compact schemes. The compact schemes provide high-order accuracy and they are also parsimonious while solving the problems on hypercube computational domains as compared to FDMs. Apart from this advantage, another exception is that compact schemes can be developed in three ways. In the first approach, original equation is considered as an auxiliary equation and each of the derivative of leading term of truncation error is compactly approximated, see [7, 8]. The second approach is known as the operator compact implicit (OCI) method. In this approach, a relationship on three adjacent points between PDE operator and unknown variable is obtained and resulting fourth order accurate relationship is derived by Taylor series expansion, see [9, 10]. In the third approach, Hermitian schemes are considered for spatial discretization of PDEs, see [11, 12]. Although compact schemes have already been proposed for option pricing in [13, 14] and in many other papers using first approach, we consider third approach for developing IMEX compact scheme in this manuscript because of the following reasons:

- It is comparatively easy to develop a compact scheme for solving high-dimensional PDEs using the third approach (see [15] for reference) as compared with the first approach (see [16] for reference).
- Recently, a fourth order accurate compact scheme is developed for space fractional advection-diffusion reaction equations with variable coefficients using the third approach in [17]. It is also explained there that first and second approaches are either not feasible or very tedious for such equations.
- It is straightforward to develop the compact scheme for the variable coefficient problems using the third approach just by discretizing the variable coefficients at each grid point, see [15, 18] for more details. However, it is cumbersome with the first approach because one has to take care of the compact discretization of the coefficient term also, see [8, 19] for detailed discussion.

In this manuscript, an IMEX compact scheme is proposed for solving the following PIDE governing the price of European options under jump-diffusion model (see [5]):

$$\begin{aligned} \frac{\partial u}{\partial \tau}(x,\tau) &= \mathbb{L}u,\ (x,\tau) \in\ (-\infty,\infty)\times(0,T], \\ u(x,0) &= f(x)\ \ \forall\ \ x\in(-\infty,\infty), \end{aligned} \tag{1}$$

where

$$\mathbb{L}u = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(x,\tau)+\left(r-\frac{\sigma^2}{2}-\lambda\zeta\right)\frac{\partial u}{\partial x}(x,\tau)-(r+\lambda)u(x,\tau)+\lambda\int_{\mathbb{R}} u(y,\tau)g(y-x)dy, \tag{2}$$

$\tau = T-t$, $x = ln\left(\frac{S}{K}\right)$, $u(x,\tau) = V(Ke^x, T-\tau)$, λ is the intensity of the jump sizes, $\zeta = \int_{\mathbb{R}}(e^x-1)g(x)dx$, and $V(S,0)$ is the option price. In this manuscript, Merton's