INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 4, Pages 557–576

doi: 10.4208/ijnam2023-1024

A PENALTY FINITE ELEMENT METHOD FOR THE STATIONARY CLOSED-LOOP GEOTHERMAL MODEL

HAOCHEN LIU, PENGZHAN HUANG*, AND YINNIAN HE

Abstract. In this article, we give a penalty finite element method for the steady-state closed-loop geothermal model. Firstly, we construct the stationary penalty closed-loop geothermal equations. Secondly, we propose a finite element method for the penalty system and deduce error estimates. Finally, some numerical experiments are used to illustrate the theoretical results of the presented method.

Key words. Closed-loop geothermal model, penalty finite element method, error estimate, numerical test.

1. Introduction

In this paper, we propose and study a penalty finite element method for a steady-state closed-loop geothermal model. The governing equations of this model include the Navier-Stokes/Darcy equations and heat equations [20, 21].

Let $\Omega \subset \mathbb{R}^2$ consist of two subdomains Ω_f and Ω_p with Lipschitz continuous boundaries $\partial\Omega_f$ and $\partial\Omega_p$, separated by the interface Γ. The Navier-Stokes equations coupled with the heat equation (velocity vector u_f, pressure p_f and temperature θ_f) describe fluid flow in the fluid domain Ω_f:

$$-\nu\Delta u_f + (u_f \cdot \nabla)\, u_f + \nabla p_f = G_r \nu^2 \theta_f \xi \qquad \text{in } \Omega_f, \tag{1}$$

$$\nabla \cdot u_f = 0 \qquad \text{in } \Omega_f, \tag{2}$$

$$-\alpha_f \Delta\theta_f + u_f \cdot \nabla\theta_f = g_f \qquad \text{in } \Omega_f. \tag{3}$$

Besides, the Darcy equations coupled with the heat equation (velocity vector u_p, pressure p_p and temperature θ_p) describe Darcy flow in the porous media domain Ω_p:

$$\frac{\nu}{D_a} u_p + \nabla p_p = G_r \nu^2 \theta_p \xi \qquad \text{in } \Omega_p, \tag{4}$$

$$\nabla \cdot u_p = 0 \qquad \text{in } \Omega_p, \tag{5}$$

$$-\alpha_p \Delta\theta_p + u_p \cdot \nabla\theta_p = g_p \qquad \text{in } \Omega_p, \tag{6}$$

where ν is the kinetic viscosity and G_r is the Grashof number, and $\xi = (0, -1)^T$ is the unit vector in the direction of the gravitational acceleration. In addition, D_a is the Darcy number, assuming the porous media is isotropic and homogeneous. α_f and α_p refer to the thermal diffusivity in the fluid and porous media domains, respectively. g_f and g_p are heat sources in the fluid and porous media domains.

Received by the editors on October 5, 2022 and, accepted on April 14, 2023.
2000 *Mathematics Subject Classification.* 65N30, 76M10, 76N10.
*Corresponding author.

Additionally, the problem (1)-(6) is considered in conjunction with the following boundary conditions on $\partial\Omega_f$ and $\partial\Omega_p$

$$u_f = 0 \text{ on } \partial\Omega_f\backslash\Gamma, \quad \theta_f = 0 \text{ on } \Gamma_{fD}, \quad \frac{\partial\theta_f}{\partial n_f} = 0 \text{ on } \Gamma_{fN},$$
$$u_p \cdot n_p = 0 \text{ on } \partial\Omega_p\backslash\Gamma, \quad \theta_p = 0 \text{ on } \Gamma_{pD}, \quad \frac{\partial\theta_p}{\partial n_p} = 0 \text{ on } \Gamma_{pN},$$

where Γ_{fD} and Γ_{fN} are the pipe region boundaries with $\partial\Omega_f\backslash\Gamma = \Gamma_{fN} \cup \Gamma_{fD}$ and denote the Dirichlet and Neumann boundary conditions, respectively. Γ_{pD} and Γ_{pN} are the porous media region boundaries with $\partial\Omega_p\backslash\Gamma = \Gamma_{pN} \cup \Gamma_{pD}$. The unit outward normal vectors satisfy the condition of $n_p = -n_f$ on the interface Γ. Furthermore, for the closed-loop geothermal model, in order to describe heat exchanging and no-fluid communication on the interface Γ, we utilize several critical interface conditions as follows [18]:

$$\theta_f = \theta_p, \qquad \text{(Continuity of temperature)}, \tag{7}$$
$$\alpha_f \frac{\partial\theta_f}{\partial n_f} + \alpha_p \frac{\partial\theta_p}{\partial n_p} = 0, \qquad \text{(Continuity of heat flux)}, \tag{8}$$
$$u_p \cdot n_p = 0, \quad u_f \cdot n_f = 0, \qquad \text{(No-communication condition)}, \tag{9}$$
$$u_f \cdot \tau = 0, \qquad \text{(No-slip condition)}, \tag{10}$$

where τ is the unit tangential vector along Γ.

Numerically solving the governing problem remains challenging because it has multiple physical quantities and domain couplings. In [24], Valencia-López et al. propose some finite element methods to study Navier-Stoke/Darcy equations coupled with heat equations, where the Navier-Stokes/Darcy equations are coupled with the Beavers-Joseph interface conditions. In [26], Zhang et al. consider the well-posedness and numerical scheme for the natural convection in a composite fluid layer overlying a porous media layer with internal heat generation. In addition, Mahbub et al. [18] use an unsteady-state model with no-communication conditions on the interface for the closed-loop geothermal system and design a decoupled stabilized finite element approach. A decoupled iterative finite element method and a two-grid finite element method [15, 14] are proposed and analyzed for the steady-state case.

Since the Darcy velocity u_p in (4) has low regularity ($u_p \in L^2(\Omega_p)^2$), it is not easy to prove the existence of a solution to the problem (1)-(10) and have a challenge in the numerical computations. As is known, the penalty method [4, 22, 23, 19, 9, 11, 12, 13, 17] is a practical algorithm for fluid flow problems. In this article, we propose and study a penalty finite element method for the steady-state closed-loop geothermal model. Firstly, we construct the stationary penalty closed-loop geothermal equations and prove the existence of the weak solution to the penalty system, which is easier to obtain than that of the original system. Then we get error estimates of the weak solutions to the penalty and the original system. Secondly, we propose a finite element method for the penalty system and deduce the convergence of the finite element discretization. Finally, since the finite element system is nonlinear, we linearize the nonlinear problem and deduce the iterative error.

Now, we use the penalty method for the original equations of the closed-loop geothermal problem. The penalty method applied to (1)-(6) is to approximate the solution ($\boldsymbol{u} = (u_f, u_p), \boldsymbol{p} = (p_f, p_p), \boldsymbol{\theta} = (\theta_f, \theta_p)$) by ($\boldsymbol{u}^\varepsilon = (u_f^\varepsilon, u_p^\varepsilon), \boldsymbol{p}^\varepsilon =$