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A NEW WEAK GALERKIN METHOD WITH WEAKLY ENFORCED DIRICHLET BOUNDARY CONDITION

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Abstract. A new weak Galerkin method with weakly enforced Dirichlet boundary condition is proposed and analyzed for the second order elliptic problems. Two penalty terms are incorporated into the weak Galerkin method to enforce the boundary condition in the weak sense. The new numerical scheme is designed by using the locally constructed weak gradient. Optimal order error estimates are established for the numerical approximation in the energy norm and usual L^2 norm. Moreover, by using the Schur complement technique, the unknowns of the numerical scheme are only defined on the boundary of each piecewise element and an effective implementation of the reduced global system is presented. Some numerical experiments are reported to demonstrate the accuracy and efficiency of the proposed method.

Key words. weak Galerkin, finite element methods, weak gradient, second order elliptic problems, polytopal partitions.

1. Introduction

This paper is focused on the new developments of the weak Galerkin (WG) method with weakly enforced Dirichlet boundary condition. For simplicity, we consider the second order elliptic problem that finds u satisfying

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^d(d=2,3)$, and the diffusion tensor $a=\{a_{ij}\}_{i,j=1}^d$ is a symmetric, uniformly positive definite matrix in $\mathbb{R}^{d\times d}$.

Various finite element methods have been proposed for the second order elliptic problems. The conforming finite element method is well known but requires the continuous piecewise polynomials on simplicial grids, which leads to the difficulty in practice. To address this difficulty, the discontinuous Galerkin methods [9], the hybrid high-order method [29], virtual element methods [1, 2] and WG methods [34, 36, 37, 47, 48] have attracted wide attention. The WG method first proposed in [34] is a natural extension of the standard finite element method through a relaxed continuity requirement for the approximating functions. Due to this weak continuity, the WG methods have some advantages such as high flexibility in both numerical approximations and mesh generations. Moreover, the WG methods are generally capable of guaranteeing the physical conservation of many problems. The WG methods have been applied to solve a wide range of PDEs including the biharmonic equation [38, 43], wave equation [15], Stokes equations [39], linear elasticity equation [20], Cahn-Hilliard equation [40], Brinkman equation [23], elliptic interface problems [24], two-phase model [13], Navier-Stokes equation [16], Maxwell's equation [25, 31]. Since then, the primal dual WG methods were proposed to simulate some challenging problems including elliptic Cauchy problem [41], second order elliptic equation in non-divergee form [11, 42], Fokker-Planck type equations [44], div-curl systems with low regularity solutions [10, 19], first order transport problems

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[46]. Recently, Ye and Zhang [49, 50] introduced the stabilizer free WG methods to simplify the standard WG method, which have been successfully applied to solve many partial differential equations (PDEs) including second order elliptic problems [33], Stokes equations [26], biharmonic equation [51], monotone quasilinear elliptic PDEs [52].

The enforcement of Dirichlet boundary conditions is crucial in practice. It is well known that the enforcement of the strong boundary conditions is not complicated only if the computational meshes perfectly match the solving domain of the model problems, while the implementation of all other cases is very complicated [7, 30]. For the simple variational problems, the strong enforced Dirichlet boundary conditions are easy to implement and can provide a numerical approximation with needed order of convergence so that the strong boundary conditions are usually employed in many numerical methods. However, it is difficult to obtain an accurate numerical approximation for some problems such as the interface problems [21] and Naiver-Stokes equation with low Reynolds number on coarse meshes [8]. To address these issues, the weakly enforced boundary conditions have been extensively studied. The most popular approaches include Nitsche's method [28], the penalty method [5] and Lagrange multiplier method [6].

The objective of this paper is to present the weak Galerkin method with weakly enforced Dirichlet boundary condition for the Possion equation (1). The idea is to weakly impose the boundary condition through the introduction of a Lagrange multiplier. Specifically, the WG form can be obtained by seeking $u_h \in V_h$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{aligned} s(u_h, v) + (a\nabla_w u_h, \nabla_w v) - \langle \lambda_h, v\rangle_{\partial\Omega} =&(f, v), \quad \forall v \in V_h, \\ h^{\alpha}\langle \lambda_h, \sigma\rangle_{\partial\Omega} + \langle \sigma, u_b\rangle_{\partial\Omega} =&\langle \sigma, g\rangle_{\partial\Omega}, \ \forall \sigma \in \Lambda_h, \end{aligned} \tag{2}$$

where a Lagrangian multiplier λ_h is given by $\lambda_h = h^{-\alpha}(-u_b + Q_b g)$, and then first equation in (2) can be rewritten as

$$s(u_h, v) + (a\nabla_w u_h, \nabla_w v) + h^{-\alpha}\langle u_h, v\rangle_{\partial\Omega} = (f, v) + h^{-\alpha}\langle g, v\rangle_{\partial\Omega}, \quad \forall v \in V_h. \tag{3}$$

Different from the penalty method [5], two additional penalty terms are designed through a new Lagrangian multiplier. In addition, the generalized weak gradient is employed in the numerical algorithm. The weak enforcement of Dirichlet boundary conditions has been incorporated into various numerical methods including finite element method [17, 27], hybrid high-order method [12], virtual element method [4], extended finite element methods[45]. To the best of our knowledge, most of the existing results on the WG framework are available in the sense of strong Dirichlet boundary conditions. One such exception is the modified WG method [14] that introduced different penalty terms.

The main novelty of this paper is the following. Firstly, the new WG method with the weak enforcement of Dirichlet boundary condition is a non-trivial generalization of the classical Nitsche method. This lays the potential foundation in solving many PDEs that are difficult to enforce the Dirichlet boundary condition in the strong sense. Secondly, our numerical method allows the general polytopal partitions which makes the structure of numerical approximations and mesh generations more flexible. Moreover, the new method can provide an easy-to-implement technique for certain boundary condition. Finally, we observe from some numerical results that the maximum principle of the WG method for the model equation (1) with low regularity solutions holds true on general polytopal meshes, for which the rigorous mathematical analysis has not been developed.