INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 5, Pages 724–738


doi: 10.4208/ijnam2023-1032

# A NOVEL DEEP NEURAL NETWORK ALGORITHM FOR THE HELMHOLTZ SCATTERING PROBLEM IN THE UNBOUNDED DOMAIN

ANDY L YANG†

**Abstract.** In this paper, we develop a novel meshless, ray-based deep neural network algorithm for solving the high-frequency Helmholtz scattering problem in the unbounded domain. While our recent work [44] designed a deep neural network method for solving the Helmholtz equation over finite bounded domains, this paper deals with the more general and difficult case of unbounded regions. By using the perfectly matched layer method, the original mathematical model in the unbounded domain is transformed into a new format of second-order system in a finite bounded domain with simple homogeneous Dirichlet boundary conditions. Compared with the Helmholtz equation in the bounded domain, the new system is equipped with variable coefficients. Then, a deep neural network algorithm is designed for the new system, where the rays in various random directions are used as the basis of the numerical solution. Various numerical examples have been carried out to demonstrate the accuracy and efficiency of the proposed numerical method. The proposed method has the advantage of easy implementation and meshless while maintaining high accuracy. *To the best of the author's knowledge, this is the first deep neural network method to solve the Helmholtz equation in the unbounded domain.*

**Key words.** Deep Learning, plane wave, deep neural network, loss, high frequency, Helmholtz equation.

## 1. Introduction

Scattering occurs when light, sound, or moving particles encounter inhomogeneities in their medium of propagation (e.g., the illuminated object has a curved or rough surface). As an illustration, a beam of light traveling through dilute milk appears pink when viewed from below, but light blue when viewed from the side and above. The study of scattering phenomena has close ties to engineering and technology, such as the use of tropospheric scattering for microwave or ultrashortwave in communication technology [22] and the investigation of the dynamical properties of atoms or electrons using scattering phenomena of various rays in the field of material science [32]. Due to the fact that scattering originates from the wave nature of matter [26, 33], its mathematical model is the standard wave equation or other equations derived from it. It is known, however, that the wave equation is a system of partial differential equations containing both time and space variables, which makes it more challenging to solve, especially when the boundary condition is complex, such as being unsmooth or unbounded. Thus, the Helmholtz equation, a simplified form of the time-independent wave equation obtained by using the separation of variables to simplify the analysis, has since become the landmark model for the study of scattering theory in its broadest sense. Moreover, if the Helmholtz equation is understood from the perspective of operator, the constant used to represent the wave frequency (or called wave number) can also be regarded as an eigenvalue, and the Helmholtz equation becomes an eigenvalue problem of the Laplace operator [13].

Received by the editors on April 17, 2023 and, accepted on September 7, 2023.
2000 *Mathematics Subject Classification.* 65L15, 34L16.

It is well known that when certain mathematical models are too complex to be solved analytically, it has become common subconscious to use computers for numerical simulations, as is the case with the wave equation and its simplified form, the Helmholtz equation. Notice that both, especially the latter, have a wide range of scientific and engineering applications, such as optics, acoustics, electricity, and quantum mechanics. For example, the cornerstone of non-relativistic quantum mechanics, Schrdinger's equation, is an extension of the Helmholtz equation, and a special case of the Helmholtz equation, the Laplace equation, also appears frequently in electrostatics. Thus, the ubiquitous application of these fundamental physical models in natural sciences, from satellite launches in space engineering or radar signal propagation in submarines to the design of cell phones or the acoustic detection of advanced materials, has driven the urgent need to design reliable and accurate numerical methods for them. This is also the purpose of this paper, namely to design reliable and accurate numerical methods for computing numerical solutions to the Helmholtz equation using computer technology.

However, not only is it not easy to simulate them numerically, but also very challenging. The main difficulty in solving the Helmholtz equation numerically comes from its non-positive definite structure, which causes a high degree of oscillation of the solution at large wave frequencies, leading to the so-called "pollution effect" [1], i.e., the approximate solution obtained in numerical calculations has only a very low accuracy. At the same time, we know that the scattering phenomenon of waves usually occurs in the unbounded region in practical physical and engineering applications. Consequently, for the Helmholtz equation over an unbounded region that is more applicable to real-world circumstances, the unbounded nature of the region poses additional obstacles to the development of effective and accurate numerical approaches. Therefore, this paper aims to address these two challenges simultaneously, i.e., designing reliable and accurate numerical methods for the Helmholtz equation with high frequency and located in the unbounded domain.

Indeed, we note that a considerable amount of work has been focused on dealing with these numerical challenges of the Helmholtz equation. The available numerical methods can be roughly divided into two main categories, namely, traditional mesh-based methods and novel meshless deep neural network (DNN) methods. In the following, we review these two categories of numerical methods for two different cases: bounded regions and unbounded regions, respectively.

For the case of bounded regions, the former traditional type of mesh-based methods for solving the Helmholtz equation includes the finite element method (FEM), Discontinuous Galerkin/ hybridizable discontinuous Galerkin/ weak Galerkin methods, and the Spectral method, etc. (see [1, 8, 25, 31, 37, 41] and reference therein), for dealing with the Helmholtz equation equipped with various boundary conditions in a finite bounded region. Due to the highly oscillatory character of the solution, higher-order polynomials or oscillatory non-polynomial basis [31, 45] are typically employed in these approaches to prevent pollution effects; however, the computational cost is significant due to the extra degrees of freedom. Using the latter type of methods, the recently prevalent DNN approach to solve the Helmholtz equation in bounded regions, has produced relatively little work to date. To name only a few that the author is aware of, existing approaches include the so-called Deep-Least Squares method (DLSM) developed in [6], the plane wave activation based neural network method (PWNN) in [42], and the ray-based DNN (RBDNN) method in our recent work [44], etc. Although the DNN method is still in its infancy in solving the Helmholtz equation, it has features that traditional methods do not have, such