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ORTHOGONAL SPLINE COLLOCATION FOR POISSON'S EQUATION WITH NEUMANN BOUNDARY CONDITIONS

BERNARD BIALECKI* AND NICK FISHER

Abstract. We apply orthogonal spline collocation with splines of degree $r \geq 3$ to solve, on the unit square, Poisson's equation with Neumann boundary conditions. We show that the H^1 norm error is of order r and explain how to compute efficiently the approximate solution using a matrix decomposition algorithm involving the solution of a symmetric generalized eigenvalue problem.

Key words. Poisson's equation, Neumann boundary conditions, orthogonal spline collocation, convergence analysis, matrix decomposition algorithm.

1. Introduction

In this paper we consider Poisson's equation

(1) $$-\Delta u = f(x_1, x_2), \quad (x_1, x_2) \in \Omega = (0,1) \times (0,1),$$

where $\Delta \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ and u satisfies nonhomogeneous Neumann boundary conditions

(2) $$u_{x_1}(\alpha, x_2) = g_1(\alpha, x_2), \quad \alpha = 0, 1, \quad x_2 \in [0,1],$$

(3) $$u_{x_2}(x_1, \beta) = g_2(x_1, \beta), \quad x_1 \in [0,1], \quad \beta = 0, 1.$$

Using (1)–(3) and integrating with respect to x_1 and x_2, we obtain

(4)
$$\int_\Omega f(x_1,x_2)dx_1dx_2 + \int_0^1 [g_1(1,x_2) - g_1(0,x_2)]dx_2 + \int_0^1 [g_2(x_1,1) - g_2(x_1,0)]dx_1 = 0,$$

which is a necessary condition for the existence of u satisfying (1)–(3). To guarantee uniqueness of the solution u of (1)–(3), we impose the condition

(5) $$\int_\Omega u(x_1,x_2)dx_1dx_2 = \gamma,$$

where γ in R is specified.

A finite difference scheme for (1)–(3) in section 4.7.2 of [12], involving an extended system of linear equations, is second order accurate in the discrete maximum norm. A finite difference scheme for (1)–(3), (5), described in Theorem 9 on page 327 in [17], involving a finite difference counterpart of (5), is second order accurate in the discrete H^1 norm. It is also shown in Theorem 2 on page 338 in [17] that this scheme is second order accurate in the discrete maximum norm. [1] is concerned with a Galerkin spectral solver for the Neumann problem for the constant coefficient Helmholtz equation on a rectangle. Finite element schemes for solving the pure Neumann problem on a bounded domain Ω are discussed in [13]. The present paper is a generalization of [4] to nonzero Neumann boundary conditions and splines of arbitrary degree $r \geq 3$. Moreover, in comparison to [4], the scheme in the present paper involves orthogonal spline collocation (OSC) counterpart of

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*Corresponding author.

(5), rather than the least squares solution. Also, using the OSC analog of the Poincaré inequality, we show that the H^1 norm of the error, rather than its H^1 seminorm considered in [4], is of order r. Hence our OSC scheme is more accurate than FD schemes of [12, 17]. The Galerkin spectral solution of [1] is obtained using the matrix decomposition algorithm that involves the computation of eigenvalues and eigenvectors of a symmetric pentadiagonal matrix. In comparison, we obtain our OSC solution using the matrix decomposition algorithm which involves finding eigenvalues and eigenvectors of a generalized symmetric banded eigenvalue problem. For $r = 3$, explicit formulas for eigenvalues and eigenvectors of these eigenvalue problems are given in [4]. Our OSC solution is required to satisfy nonzero Neumann boundary conditions at corners and collocation points on $\partial\Omega$ while in [1], a function, defined on $\overline{\Omega}$ and satisfying nonzero Neumann boundary conditions on $\partial\Omega$, is first determined. Spectral accuracy of the approximate solution in [1] is demonstrated numerically only while we give a theoretical convergence analysis of our OSC scheme. In [1], integrals involving f of (4) are evaluated approximately which is unnecessary for our OSC scheme. Generalization of our OSC scheme to (1)–(3) with (1) replaced by the separable equation

$$\sum_{i=1}^{2}[-a_i(x_i)u_{x_ix_i} + c_i(x_i)u] = f(x_1, x_2), \quad (x_1, x_2) \in \Omega, \tag{6}$$

with variable coefficients

$$a_i(x_i) > 0, \quad c_i(x_i) \geq 0, \quad x_i \in [0, 1],$$

still involves solution of a generalized symmetric banded eigenvalue problem, while the approach in [1] for (6), with $-(a_iu_{x_i})_{x_i}$ replacing $-a_i(x_i)u_{x_ix_i}$, involves solution of a generalized symmetric eigenvalue problem with full matrices. Like scheme in [1], our scheme generalizes to 3 dimensions, in which case the cost of solving eigenvalue problem is negligible in comparison to the total cost of the solution process. Matrix decomposition algorithms for solving finite element Galerkin schemes for separable equations on a rectangle were developed in [14, 15] for Dirichlet and mixed boundary conditions. However, [14, 15] do not provide details of such algorithms for solving a singular linear system arising in the case of the pure Neumann problem (1)–(3). The OSC solution of the Neumann problem on a rectangle was recently used in a pressure Poisson OSC method for solving the Navier-Stokes equation [11].

The paper is outlined as follows. Section 2 gives some preliminary results used in the convergence analysis. Section 3 introduces an OSC solution for (1)–(3), (5). Error bounds are derived in Section 4. A matrix decomposition algorithm to find the OSC solution is described in Section 5. Finally, numerical results are presented in Section 6.

2. Preliminaries

In what follows, $\delta_{x_1} = \{x_1^{(i)}\}_{i=0}^{N_{x_1}}$ and $\delta_{x_2} = \{x_2^{(j)}\}_{j=0}^{N_{x_2}}$ are partitions of $[0, 1]$, such that,

$$0 = x_1^{(0)} < x_1^{(1)} < \cdots < x_1^{(N_{x_1})} = 1, \quad 0 = x_2^{(0)} < x_2^{(1)} < \cdots < x_2^{(N_{x_2})} = 1,$$

and $\delta = \delta_{x_1} \times \delta_{x_2}$. We introduce

$$h_i^{x_1} = x_1^{(i)} - x_1^{(i-1)}, \quad i = 1, \ldots, N_{x_1}, \quad h_j^{x_2} = x_2^{(j)} - x_2^{(j-1)}, \quad j = 1, \ldots, N_{x_2},$$

and we set

$$h_{x_1} = \max_{i=1,\ldots,N_{x_1}} h_i^{x_1}, \quad h_{x_2} = \max_{j=1,\ldots,N_{x_2}} h_j^{x_2}, \quad h = \max(h_{x_1}, h_{x_2}).$$