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HIGH-ORDER ENRICHED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS WITH DISCONTINUOUS SOLUTIONS

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Abstract. Elliptic interface problems whose solutions are C^0 continuous have been well studied over the past two decades. The well-known numerical methods include the strongly stable generalized finite element method (SGFEM) and immersed FEM (IFEM). In this paper, we study numerically a larger class of elliptic interface problems where their solutions are discontinuous. A direct application of these existing methods fails immediately as the approximate solution is in a larger space that covers discontinuous functions. We propose a class of high-order enriched unfitted FEMs to solve these problems with implicit or Robin-type interface jump conditions. We design new enrichment functions that capture the imposed discontinuity of the solution while keeping the condition number from fast growth. A linear enriched method in 1D was recently developed using one enrichment function and we generalized it to an arbitrary degree using two simple discontinuous one-sided enrichment functions. The natural tensor product extension to the 2D case is demonstrated. Optimal order convergence in the L^2 and broken H^1-norms are established. We also establish superconvergence at all discretization nodes (including exact nodal values in special cases). Numerical examples are provided to confirm the theory. Finally, to prove the efficiency of the method for practical problems, the enriched linear, quadratic, and cubic elements are applied to a multi-layer wall model for drug-eluting stents in which zero-flux jump conditions and implicit concentration interface conditions are both present.

Key words. Generalized finite element method, elliptic interface, implicit interface jump condition, Robin interface jump condition, linear and quadratic finite elements.

1. Introduction

Consider the interface two-point boundary value problem

$$\begin{cases} -(\beta(x)u'(x))' + w(x)u(x) = f(x), & x \in (a,\alpha)\cup(\alpha,b), \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $w(x) \geq 0$, and $0 < \beta \in C[a,\alpha]\cup C[\alpha,b]$ is discontinuous across the interface α with the jump conditions on u and its flux $q := -\beta u'$:

$$[u]_\alpha = \lambda F(q^+, q^-), \quad \lambda \in \mathbb{R},\, F : [c_1, d_1]^2 \to \mathbb{R}, \tag{2}$$

$$[\beta u']_\alpha = f_\alpha, \quad f_\alpha \in \mathbb{R}, \tag{3}$$

where the jump quantity

$$[s]_\alpha := s(\alpha^+) - s(\alpha^-), \quad s^\pm := s(\alpha^\pm) := \lim_{\epsilon\to 0^+} s(\alpha \pm \epsilon).$$

The primary variable u may stand for the pressure, temperature, or concentration in a medium with certain physical properties and the derived quantity $q := -\beta u'$ is the corresponding Darcy velocity, heat flux, or concentration flux, which is equally important. The piecewise continuous β reflects a nonuniform material or medium property (we do not require β to be piecewise constant). The function $w(x)$ reflects the surroundings of the medium characterising the coefficient of the reaction term. The case of $\lambda = 0$ is widely studied, while the case of $\lambda > 0$ gives rise to a more

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difficult situation. For example, the case of rightward concentration flow [29, 30, 31] imposes

$$\begin{cases} [u]_\alpha &= \lambda(\beta u')(\alpha^-) \\ [\beta u']_\alpha &= 0, \end{cases} \tag{4}$$

which generates an implicit condition since the left-sided derivative is unknown. Implicit interface conditions abound in higher dimensional applications [1, 16, 19, 21]. For definiteness, we will study a class of efficient enriched methods for problem (1) under the jump conditions (4),

The model problem (1), allowing the solution to be discontinuous at the interfaces, is a more general form of interface problem than the ones studied by Babuška *et. al.* using SGFEM [4, 3, 20] and by Li *et.al.* using immersed finite element method (IFEM) [22, 23, 25, 24]. In these works, the interface problem is assumed to have a continuous solution at the interfaces. For the case of the discontinuous solution at the interfaces, some IFE methods have also been developed, see for example [35, 36]. A large class of the methods is developed based on the unfitted meshes, which has been demonstrated to be more efficient than methods using fitted meshes, especially when the interface is moving [17]; see also [7, 10, 14, 15, 18, 29].

The generalized FEMs (GFEM) were first introduced to capture certain known features of the solution to the crack problem [5, 6, 13, 26]. However, it has been shown that GFEM suffers from a lack of robustness with respect to mesh configurations and bad conditioning. In general, the condition numbers of GFEM can grow with an order $\mathcal{O}(h^{-4})$ where h characterizes the mesh size. This is two orders of magnitude worse than the standard FEM (known to be of order $\mathcal{O}(h^{-2})$). To resolve this issue, SGFEM has been developed [3, 4, 20] with an extra feature of robustness with respect to the mesh configurations. Further development of SGFEM has been active. For example, [34] extends the linear SGFEM to quadratic-order; [12] extends the method for eigenvalue interface problems, and [33] generalizes the SGFEM idea to isogeometric analysis with B-spline basis functions.

In 1D, when the solution to the underlying interface problem is continuous, a single enrichment function associated with the interface can be used for arbitrary high-order elements in the SGFEM. However, for an interface problem whose solution is discontinuous at interfaces, the natural extension with one enrichment function at each interface fails. A more sophisticated construction of enrichment functions is desired especially for high-order elements. This motivates the present work.

In our enriched FEM (might also call it as a GFEM), the approximation finite element space V_h^{enr} takes the form:

$$V_h^{enr} := S_h + V_E = \{p_h + q_h\psi : \, p_h, q_h \in S_h, \psi \in F_{enr}\} \tag{5}$$

where S_h is a standard finite element space (e.g., $\mathbb{P}_k$-conforming, $k \geq 1$),

$$V_E = \{q_h\psi : \, q_h \in S_h, \psi \in F_{enr}\}, \tag{6}$$

and the function ψ is from the enrichment function space

$$F_{enr} := \operatorname{span}\{\psi_0, \psi_1, \ldots, \psi_m\}, \ \dim(F_{enr}) = m + 1. \tag{7}$$

Here the basis functions ψ_i capture the interface condition(s) at α, e.g., zero or nonzero jump of the function value across α. For example, for a continuous solution case, a single ($m = 0$) enrichment function suffices, whereas we show in this paper that for discontinuous solution case, we need two enrichment functions ($m = 1$) defined in (10). There are some distinct features about V_h^{enr} in this case.