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A PARALLEL ITERATIVE PROCEDURE FOR WEAK GALERKIN METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. A parallelizable iterative procedure based on domain decomposition is presented and analyzed for weak Galerkin finite element methods for second order elliptic equations. The convergence analysis is established for the decomposition of the domain into individual elements associated to the weak Galerkin methods or into larger subdomains. A series of numerical tests are illustrated to verify the theory developed in this paper.

Key words. Weak Galerkin, finite element methods, elliptic equation, parallelizable iterative, domain decomposition.

1. Introduction

This paper is concerned with an iterative procedure related to domain decomposition techniques based on the use of subdomains as small as individual elements for weak Galerkin (WG) methods for second order elliptic equations in $\mathbb{R}^d (d = 2, 3)$. For simplicity, we consider the second order elliptic problem with a Dirichlet boundary condition

$$
\begin{aligned}
-\nabla \cdot (a\nabla u) + cu =& f, \qquad \text{in } \Omega \subset \mathbb{R}^d, \\
u =& g, \qquad \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where $d = 2, 3$. Assume the coefficients $a(x)$ and $c(x)$ satisfy

$$0 < a_0 \le a(x) \le a_1 < \infty, \quad 0 \le c(x) \le c_1 < \infty,$$

and are sufficiently regular so that the existence and uniqueness of a solution of (1) in $H^s(\Omega)$ hold true for some $s > 1$ for reasonable f and g. A weak formulation for the model problem (1) reads as follows: Find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$, satisfying

$$(a\nabla u, \nabla v) + (cu, v) = (f, v) \qquad \forall v \in H_0^1(\Omega). \tag{2}$$

The WG finite element method is emerging as a new and efficient numerical method for solving partial differential equations (PDEs). The idea of WG method was first proposed by Wang and Ye for solving second order elliptic equations in 2011 [33]. This method was subsequently developed for various PDEs, see [15, 16, 18, 19, 20, 31, 27, 30, 28, 26, 31, 33, 34, 17, 35, 36, 27, 29, 32]. Due to the large size of the computational problem, it is necessary and crucial to design efficient and parallelizable iterative algorithms for the WG scheme. There have been some iterative algorithms designed for the WG methods along the line of domain decompositions [23, 5, 22, 21, 14]. Our iterative procedure is motivated by Despres [6] for a Helmholz problem and another Helmholz-like problem related to Maxwell's equations by Despres [7, 8]. It should be noted that the convergence in [6, 7, 8]

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were established for the differential problems in strong form where numerical results were presented to validate the iterative procedures for the discrete case. Douglas et al. [9] introduced a parallel iterative procedure for the second order partial differential equations by using the mixed finite element methods. The goal of this paper is to extend the result of Douglas into weak Galerkin finite element methods. In particular, based on the features of weak Galerkin methods, the iterative procedure developed in this paper can be very naturally and easily implemented on a massively parallel computer by assigning each subdomain to its own processor.

The paper is organized as follows. In Section 2, we briefly review the weak differential operators and their discrete analogies. In Section 3, we describe the WG method for the model problem (1). In Section 4, we introduce domain decompositions and derive a hybridized formulation for the WG method. In Section 5, we present a parallel iterative procedure for the WG finite element method. In Section 6, we establish a convergence analysis for the parallel iterative scheme. Finally in Section 7, we report several numerical results to verify our convergence theory.

2. Weak Differential Operators

The primary differential operator in the weak formulation (2) for the second order elliptic problem (1) is the gradient operator ∇, for which a discrete weak version has been introduced in [34]. For completeness, let us briefly review the definition as follows.

Let T be a polygonal or polyhedral domain with boundary ∂T. A weak function on T refers to $v = \{v_0, v_b\}$ where $v_0 \in L^2(T)$ and $v_b \in L^2(\partial T)$ represent the values of v in the interior and on the boundary of T respectively. Note that v_b may not necessarily be the trace of v_0 on ∂T. Denote by $W(T)$ the local space of weak functions on T; i.e.,

$$W(T) = \{v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in L^2(\partial T)\}.$$

The weak gradient of $v \in W(T)$, denoted by $\nabla_w v$, is defined as a linear functional on $[H^1(T)]^d$ such that

$$(\nabla_w v, \mathbf{w})_T = -(v_0, \nabla \cdot \mathbf{w})_T + \langle v_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T} \qquad \forall \mathbf{w} \in [H^1(T)]^d.$$

Denote by $P_r(T)$ the space of all polynomials on T with total degree r and/or less. A discrete version of $\nabla_w v$ for $v \in W(T)$, denoted by $\nabla_{w,r,T} v$, is defined as a unique polynomial vector in $[P_r(T)]^d$ satisfying

$$(\nabla_{w,r,T} v, \mathbf{w})_T = -(v_0, \nabla \cdot \mathbf{w})_T + \langle v_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{w} \in [P_r(T)]^d. \tag{3}$$

3. Weak Galerkin Algorithm

Let $\mathcal{T}_h$ be a finite element partition of the domain Ω consisting of polygons or polyhedra that are shape-regular [34]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or flat faces. Denote by h_T the meshsize of $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize for the partition $\mathcal{T}_h$.

For any given integer $k \geq 1$, denote by $W_k(T)$ the local discrete space of the weak functions given by

$$W_k(T) = \{\{v_0, v_b\} : v_0 \in P_k(T), v_b \in P_{k-1}(e), e \subset \partial T\}. \tag{4}$$

Patching the local discrete space $W_k(T)$ with a single value on the element interface yields the global finite element space; i.e.,

$$W_h = \{v = \{v_0, v_b\} : \ v|_T \in W_k(T), v_b \text{ is single-valued on } e \subset \mathcal{E}_h^0, T \in \mathcal{T}_h\}.$$