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CONTINUOUS/DISCONTINUOUS FINITE ELEMENT APPROXIMATION OF A 2D NAVIER-STOKES PROBLEM ARISING IN FLUID CONFINEMENT

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Abstract. In this work, a stationary 2d Navier-Stokes problem with nonlinear feedback forces field is considered in the stream-function formulation. We use the Continuous/Discontinuous Finite Element Method (CD-FEM), with interior penalty terms, to numerically solve the associated boundary-value problem. For the associated continuous and discrete problems, we prove the existence of weak solutions and establish the conditions for their uniqueness. Consistency, stability and convergence of the method are also shown analytically. To validate the numerical model regarding its applicability and robustness, several test cases are carried out.

Key words. 2d Navier-Stokes, feedback forces, CD-FEM, existence and uniqueness, consistency and stability, error analysis.

1. Introduction

Given a bounded domain $\Omega := (0,K)\times(0,L)$ of $\mathbb{R}^2$, where L and K are positive constants, let us consider the following problem for the Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f}(\mathbf{x},\mathbf{u}) - \nabla p \quad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{u}_* \quad \text{on} \quad x = 0, \tag{3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad y = 0,\ L \quad \text{and on} \quad x = K. \tag{4}$$

Here, $\mathbf{u} = (u, v)$ denotes the velocity field, p accounts for the pressure, ν is the kinematics viscosity, $\mathbf{f} = (f_1, f_2)$ is a feedback forces field (divided by the constant density ρ that is supposed to be $\rho = 1$), $\mathbf{u}_* := (u_*, v_*)$ is the prescribed velocity at the strip entrance $x = 0$, and by $\mathbf{x} = (x, y)$ we denote a generic element of $\mathbb{R}^2$.

Problem (1)-(4) can be used to describe a planar steady flow of a viscous fluid that is controlled by a feedback forces field. This type of forces field plays a central role, for instance, in the confinement of magnetic fluids in Tokamaks and Stellarators (ball- or torus-shaped devices used to produce controlled thermonuclear fusion energy). In these devices, a powerful magnetic field is used to confine very hot plasmas far enough away from the boundaries to prevent damage. The governing equations of this real-world problem consist of the Navier-Stokes equations coupled to Maxwell's equations via the Lorentz force – the feedback forces field [16]. For the sake of mathematical simplification, we only consider the Navier-Stokes problem (1)-(4) and proceed to characterize the type of forces field that can confine the fluid. The fluid confinement property we are interested in for the problem (1)-(4) can be read mathematically as follows:

$$\exists\ a \in (0, K)\ :\ \mathbf{u} = \mathbf{0} \quad \text{a.e. in } \Omega_a := (a, K)\times(0, L). \tag{5}$$

In the works [3, 4, 5, 6] we undertook a project to characterize the nature of the forces field that can confine a fluid governed by a system of equations of the type

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(1)-(4). There we look at forces fields, the notation of which in its simplest form is:

$$\mathbf{f}(\mathbf{x},\mathbf{u}) := -\delta\left(|u|^{\sigma-2}u, 0\right), \tag{6}$$

where $\delta > 0$ is a constant that accounts for the intensity of the forces field, and $\sigma > 1$ is another constant that characterizes the flow. The presence of feedback forces field of the type (6) can also be justified in other applications, such as in porous media flows and in continuous electromagnetic media. In porous media it is known as the Forchheimer term and it is important to characterize the flow resistance created by the skeleton of the porous medium, specially when the pore Reynolds number exceeds 10 [29]. This term is also considered for some quasi-stationary processes, in crystalline semiconductors, to model the density of sources or sinks of free electrons in the semiconductor lattice [2].

For the different flow conditions considered in [3, 4, 5, 6], we have shown that a feedback forces field of the type (6) can confine the fluid flow as long as the exponent of nonlinearity σ satisfies

$$1 < \sigma < 2. \tag{7}$$

The fluid confinement property was proved analytically in [3, 4, 5, 6] by considering the problem (1)-(4) in the stream-function formulation. This formulation is obtained by taking the curl of the momentum equation (1), where the forces field is given by (6), and by introducing the stream function

$$\psi \; : \; u = \psi_y \quad \text{and} \quad v = -\psi_x \quad \text{in } \Omega, \tag{8}$$

which exists in view of (2) [17, Theorem I.3.1]. By this procedure, we obtain the following fourth-order nonlinear boundary-value problem

$$\nu\Delta^2\psi + J(\psi,\Delta\psi) = \delta\left(|\psi_y|^{\sigma-2}\psi_y\right)_y \qquad \text{in} \quad \Omega, \tag{9}$$

$$\psi = f_* \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = g_* \quad \text{on } \overline{\Gamma}^*, \tag{10}$$

$$\psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\mathbf{n}} = 0 \quad \text{on } \overline{\Gamma}^0, \tag{11}$$

where $\Delta^2\psi$ accounts for the bi-Laplacian of ψ, $J(\psi,\Delta\psi)$ denotes the Jacobian

$$J(\psi,\Delta\psi) := \det\begin{bmatrix}\psi_x & \psi_y \\ \Delta\psi_x & \Delta\psi_y\end{bmatrix} = \psi_x\Delta\psi_y - \psi_y\Delta\psi_x,$$

$\frac{\partial\psi}{\partial\mathbf{n}}$ is the normal derivative of ψ, with $\mathbf{n} = (n_1, n_2)$ standing for the outward unit normal to the boundary $\partial\Omega$ of the domain Ω, which in turn is decomposed into the following two disjoint subsets

$$\overline{\Gamma}^* := \{(x,y) \in [0,K]\times[0,L] : x = 0\}, \tag{12}$$

$$\overline{\Gamma}^0 := \{(x,y) \in (0,K]\times[0,L] : y = 0 \vee y = L \vee x = K\}. \tag{13}$$

Data f_* and g_*, given in (10), can be written in terms of the prescribed velocity (u_*, v_*) at the strip entrance $x = 0$ as follows

$$f_*(y) = \int_0^y u_*(s)\,\mathrm{d}s, \qquad g_*(y) = v_*(y), \qquad y \in (0,L), \tag{14}$$

and are assumed to satisfy the following compatibility conditions

$$f_*(0) = f_*(L) = 0, \qquad g_*(0) = g_*(L) = 0. \tag{15}$$