INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 504–527

doi: 10.4208/ijnam2024-1020

MIXED VIRTUAL ELEMENT METHOD FOR LINEAR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article develops and analyses a mixed virtual element scheme for the spatial discretization of linear parabolic integro-differential equations (PIDEs) combined with backward Euler's temporal discretization approach. The introduction of mixed Ritz-Volterra projection significantly helps in managing the integral terms, yielding optimal convergence of order $O(h^{k+1})$ for the two unknowns $p(\boldsymbol{x}, t)$ and $\boldsymbol{\sigma}(\boldsymbol{x}, t)$. In addition, a step-by-step analysis is proposed for the super convergence of the discrete solution of order $O(h^{k+2})$. The fully discrete case has also been analyzed and discussed to achieve $O(\tau)$ in time. Several computational experiments are discussed to validate the proposed schemes computational efficiency and support the theoretical conclusions.

Key words. Mixed virtual element method, parabolic integro-differential equation, error estimates, super-convergence.

1. Introduction

Mathematical models for solving the electrical circuit problems specified by the Kirchhoff voltage laws [34], for a disease transmitted through the movement of contagious individuals [29], heat flow in a substance with memory [30], etc., give rise to the linear integro-differential equations. With consideration for the diverse array of applications of these equations across various domains, our focus lies in the exploration of PIDEs of the following form:

(1)
$$
\begin{aligned}
p_t(\boldsymbol{x}, t) - \nabla \cdot \left(a(x) \nabla p(\boldsymbol{x}, t) - \int_0^t b(x; t, s) \nabla p(\boldsymbol{x}, s) ds \right) &= f(\boldsymbol{x}, t) \; (\boldsymbol{x}, t) \in \mathcal{D} \times (0, T], \\
p(\boldsymbol{x}, t) = 0 \quad (\boldsymbol{x}, t) &\in \partial\mathcal{D} \times (0, T], \\
p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} &\in \mathcal{D}.
\end{aligned}
$$

Here, $\mathcal{D} \subset \mathbb{R}^2$ is a bounded polygon domain having $\partial\mathcal{D}$ as the boundary; furthermore, the interval $[0, T]$ represents a finite time span. This article intends to introduce and examine the mixed virtual element method (VEM) concerning PIDEs (1), with the primary goal of studying the effect of time discretization on virtual element solution. For our analysis, we would require the assumptions listed below on the coefficients and the function f:

H.1 the coefficient $a(x)$ is bounded, positive i.e. $a(x) \geq \mu_0 > 0$, and smooth enough,
H.2 the coefficient $b(x; t, s)$ and its derivative $b_t(x; t, s)$, $b_s(x; t, s)$ are real-valued, bounded, and smooth,
H.3 the function f is real-valued and smooth enough.

In literature, various approaches have been made to obtain the numerical solution to these equations and related problems, such as the finite element method (FEM) [23, 14], finite volume method [16], VEM [40], least-square Galerkin method [24], *hp*-local discontinuous Galerkin method [32], spectral method [19], HDG method

Received by the editors on September 22, 2023 and, accepted on April 21, 2024.
2000 *Mathematics Subject Classification.* 65M15, 65N12, 65N15.

[26] etc. Further, by extending these ideas in [31, 12, 20], fully discrete schemes were proposed in which discretization of time is implemented using implicit finite difference schemes. The reason for employing finite element scheme and their variants is the computational efficiency and well-established theory of these methods. We stress that polygonal meshes have many benefits: greater flexibility in the meshing of arbitrary geometries, better accuracy in the numerical solution over that obtainable using triangular and quadrilateral meshes on a given nodal grid, and many more, see [36]. To deal with polygonal meshes, VEM was introduced in [1] and is very much appreciated by the scientific community. A detailed study shows that this method can be considered as a generalization of the standard FEM over general polygonal and polyhedral meshes as the convergence analysis of this method can be placed within the structure of the FEM, which is well developed in the literature. In general, VEM has been successfully applied for an approximate estimation of various partial differential equations; for recent developments and applications of this method, we refer to [38, 39, 6, 2, 38] and references within.

One of our concerns in (1) is determining the flux or velocity in addition to the pressure; the typical Galerkin method yields a loss of precision because it is estimated from the approximated solution via post-processing. The mixed methods, on the other hand, provide a direct estimate of this physical quantity and lead to locally conservative solutions. Another advantage of using a mixed technique here is the ability to introduce one more unknown of physical importance, which may be computed directly without adding any new sources of error. Mixed VEM has been effectively employed to approximate a number of partial differential equations; see [9, 8, 10, 21, 11, 22] and references therein for details. Here, we introduce $\boldsymbol{\sigma}(\boldsymbol{x}, t)$, defined by

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = a(x)\nabla p(\boldsymbol{x}, t) - \int_0^t b(x; t, s)\nabla p(\boldsymbol{x}, s)ds, \tag{2}$$

and rewrite (1) as:

$$p_t(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) = f(\boldsymbol{x}, t). \tag{3}$$

The meaning of this independent variable '$\boldsymbol{\sigma}$' is velocity field while discussing flow in porous media, whereas (3) expresses a mass balance in any subdomain of $\mathcal{D}$, see[35]. So, the mixed formulation for this setting simultaneously approximates the pressure and the velocity field while maintaining the underlying local mass conservation. Since there is an integral term in (2) which involves ∇p, we introduce a new kernel known as the resolvent kernel to deal with this integral term. This formulation has been explored in [35, 17, 18] for the semi-discrete formulation and non-smooth initial data, but the fully-discrete case has not been explored yet for this formulation to the best of our knowledge.

For the mixed variational formulation corresponding to (2), we will use the resolvent kernel, applicable to any Volterra integral equation of the second kind [33], which takes the form:

$$X(t) = F(t) + \int_0^t K(t, s)X(s)ds. \tag{4}$$

The resolvent kernel for (4) can be expressed as:

$$X(t) = F(t) + \int_0^t R(t, s)F(s)ds.$$