INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 587–608

doi: 10.4208/ijnam2024-1023

THE WEAK GALERKIN FINITE ELEMENT METHOD FOR THE DUAL-POROSITY-STOKES MODEL

LIN YANG, WEI MU*, HUI PENG, AND XIULI WANG

Abstract. In this paper, we introduce a weak Galerkin finite element method for the dual-porosity-Stokes model. The dual-porosity-Stokes model couples the dual-porosity equations with the Stokes equations through four interface conditions. In this method, we define several weak Galerkin finite element spaces and weak differential operators. We provide the weak Galerkin scheme for the model, and establish the well-posedness of the numerical scheme. The optimal convergence orders of errors in the energy norm are derived. Finally, we verify the effectiveness of the numerical method with different weak Galerkin elements on different meshes.

Key words. Dual-porosity-Stokes model, weak Galerkin finite element method, discrete weak gradient, discrete weak divergence.

1. Introduction

In practical production and daily life, the coupling models of porous media flow and free flow are widely used, such as groundwater system [16], industrial filtration [13], oil exploitation [8], and biochemical transportation [11], etc. The classic Stokes-Darcy model is used frequently in the coupling flow problems [3, 5, 6, 12, 20]. However, Darcy's law is only available for single-pore model and cannot accurately describe complex porous medium model with multiple porosities, which arises, such as the hydrology and geothermal systems. Therefore, Hou et al. proposed a dual-porosity-Stokes model in [17], the authors used the matrix pressure equation to characterize the flow in the matrix medium and the microfracture pressure equation in the microfractures medium, respectively. The free flow in conduits and microfractures are governed by Stokes equations. For appropriate coupling, four physical conditions are imposed on the interface: the no-exchange condition, mass conservation condition, force balance condition, and the Beavers-Joseph-Saffman (BJS) condition.

Some efforts have been made to numerically solve the dual-porosity-Stokes model. In [1], Al Mahbub et al. proposed and analyzed two stabilized mixed finite element methods for the nonstationary dual-porosity-Stokes model: the coupled method in traditional formulation and the decoupled method based on the partition time stepping method. Then, in [2], they developed a stabilized mixed finite element method for the stationary dual-porosity-Stokes model. This method only needs to add a mesh-dependent stabilization term to ensure the numerical stability of the algorithm and does not introduce any Lagrange multiplier. Combining the IPDG method and mixed finite element method, Wen et al. [37] designed a monolithic scheme with strong mass conservation for the stationary coupled model. Gao et al. considered the Navier-Stokes equation in the free flow region to couple the microfracture-matrix system in [14]. Yang et al gave a prior estimate of the discrete solutions to the stationary dual-porosity Navier-Stokes model by constructing an

Received by the editors on January 16, 2024 and, accepted on May 14, 2024.
2000 *Mathematics Subject Classification.* 65M60, 65M12, 65M15, 35M10, 35Q35, 76D07, 76S05.
*Corresponding author.

auxiliary problem and proved the existence and uniqueness of the discrete solutions in [38].

In this paper, we introduce the weak Galerkin (WG) finite element method for dual-porosity-Stokes model. The WG method was proposed by Wang and Ye in [31] for solving the second-order elliptic problem. The key idea of this method is that the solutions are approximated by discontinuous weak functions and the classical derivative operators in variational formulation are replaced by weakly defined derivative operators. At present it has many applications including parabolic equation [21, 40], Darcy equation [24, 25], Stokes equations [28, 33], Brinkman equations [26, 39], linear elasticity equations [35, 36], and so on.

For the coupled problem, as far as we know, there is some work for Stokes-Darcy model. In [9], WG finite element discretization was constructed for the Stokes equations with symmetric stress tensor and the Darcy equation in the mixed formulation. In [22, 23], the Stokes equations coupled with the Darcy equation in the primal formulation were investigated. The model is discretized by piecewise constants in [23] and high order polynomials in [22], yielding stable numerical schemes with optimal error estimates. Some methods combining the WG elements with other finite elements are discussed in [15, 29, 30].

The dual-porosity-Stokes model consists of two second-order elliptic equations in the dual-porosity domain and Stokes equation in the free flow region. The existing work has verified the efficiency of the individual Stokes equations and elliptic equations. Therefore, in this paper, we develop the WG method for the coupled model. We establish the stability of the WG scheme and prove the existence and uniqueness of the numerical solutions. Furthermore, the optimal convergence orders for the errors are obtained. The results of numerical experiments are consistent with the theoretical analysis.

The rest of the paper is organized as follows. In Section 2, we introduce the dual-porosity-Stokes model and present its variational form. In Section 3, some definitions of the weak Galerkin finite element spaces are given and then the WG numerical scheme for the coupled model is established. In Section 4, we prove the existence and uniqueness of the WG numerical solutions. In Section 5, the error equations and the corresponding optimal order error estimates are obtained. Finally, in Section 6, we present some numerical examples to verify the effectiveness of the WG method.

2. Preliminaries

In this section, we introduce the dual-porosity-Stokes model and present the corresponding variational formulation.

2.1. Dual-porosity-Stokes Model. Let Ω be a bounded domain in $\mathbb{R}^N$, $(N = 2, 3)$, which is divided into two subdomains, the dual-porosity domain Ω_d and the conduit domain Ω_c (see Figure 1). Let $\Gamma = \partial\Omega_c \cap \partial\Omega_d$ be the interface between two subdomains. Denote the boundaries of Ω_d and Ω_c by $\Gamma_d = \partial\Omega_d \backslash \Gamma$ and $\Gamma_c = \partial\Omega_c \backslash \Gamma$, respectively. In addition, $\mathbf{n}_{cd}$ is the unit normal vector on Γ which points from Ω_c into Ω_d and $\boldsymbol{\tau}_j,\ j = 1, 2, ..., N-1$ are unit tangent vectors on Γ.

The flow in the dual-porosity domain Ω_d is governed by the traditional dual-porosity model [18], which consists of matrix equation and microfracture equation.